Fairly Dividing a Cake after Some Parts were Burnt in the Oven

Erel Segal-Halevi
Dividing a heterogeneous resource to agents with different preferences such that everyone’s share is “fair” by their preferences.
Fair Division — Then

Dividing a **heterogeneous** resource to agents with **different** preferences such that everyone’s share is "fair" by their preferences.

Steinhaus | Banach | Knaster | Dubins | Spanier
Dividing a heterogeneous resource to agents with different preferences such that everyone’s share is "fair" by their preferences.

http://fairoutcomes.com

http://spliddit.org

https://math.hmc.edu/su/fairdivision

Francis Su's Fair Division Page

Click on The Fair Division Calculator which has recently been updated! (version 3.01, 4/12/00)

A java applet for interactive decision making to find envy-free divisions of goods, burdens, or rent.
Dividing a heterogeneous resource to agents with different preferences such that everyone’s share is “fair” by their preferences.
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Continuous Resource
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Cake = Interval $[0,1]$. $n$ agents. Value-densities $\nu_i : Cake \rightarrow \mathbb{R}$

Value = integral:

$$V_i(X_i) = \int_{X_i} \nu_i(x) \, dx$$
Continuous Resource

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For all $i$: $X_i$ is connected.
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Easy for 2 agents. Difficult for 3 or more.
Valuation types

All positive - solved by Stromquist (1980), Simmons (1980)
Valuation types

All positive - solved by Stromquist (1980), Simmons (1980)

All negative – solved by Su (1999)
Valuation types

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All negative – solved by Su (1999)

General – this work.
Simplex of Partitions – Definition
(based on Stromquist 1980)

Partition for 3 agents:
\[(l_1, l_2, l_3)\]
\[l_1 + l_2 + l_3 = 1\]
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Envy-free division =

\[F_1 = (1, 0, 0)\]
\[F_2 = (0.1, 0)\]
\[F_3 = (0, 0.1)\]
**Simplex of Partitions – Definition**
(based on Stromquist 1980)

Partition for 3 agents:
\[ (l_1, l_2, l_3) \]
\[ l_1 + l_2 + l_3 = 1 \]

**Envy-free division = point in which each agent prefers a different piece.**
Simplex of Partitions – Triangulation
(based on Simmons 1980, Su 1999)

a. Triangulate the simplex.
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b. Assign each vertex to a different agent such that in each sub-simplex, all agents are represented.
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b. Assign each vertex to a different agent such that in each sub-simplex, all agents are represented.

c. Ask each agent to label all its vertices by the index of his favorite piece.

d. A simplex labeled by all $n$ labels = an approximately-envy-free division.
Fact: When all agents have positive valuations, each face is labeled only with the labels of its endpoints.
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Corollary: when all valuations are positive, an approximately-envy-free division exists.
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**Lemma** (Sperner 1929): When each face is labeled only with the labels of its endpoints, a fully-labeled sub-simplex exists.

**Corollary:** when all valuations are positive, an approximately-envy-free division exists.

**Corollary** (Stromquist 1980, Simmons 1980, Su 1999): when valuations are also continuous, an envy-free division exists.
**Fact:** When all agents have negative valuations, it is possible to label the $n$ main vertices such that each face is labeled only with the labels of its endpoints.
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**Corollary:** When all valuations are negative, an approximately-envy-free division exists.
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**Corollary:** when all valuations are negative, an approximately-envy-free division exists.

**Corollary (Su 1999):** when valuations are also continuous, an envy-free division exists.
In general, the conditions for Sperner’s lemma are **not** satisfied.

What can we do?
Boundary Permutation Condition
Definition: Two vertices in the simplex are called *friends* if they have the same ordered list of non-zero coordinates.
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Fact: Each agent’s labelings on friends are same up to permutation:
Boundary Permutation Condition

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<table>
<thead>
<tr>
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<th>Right</th>
<th>Empty</th>
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</thead>
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<tr>
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![Diagram showing friends and labelings](image)
Degree of Labeling

Labeling ≡ mapping from triangulation vertices to vertices of $Q$ (follows Musin 2014)
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Labeling $\equiv$ mapping from triangulation vertices to vertices of $Q$ (follows Musin 2014)

Degree of mapping $=$ net number of rounds ($CCW=positive$).
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(follows Musin 2014)

Degree of Labeling

Degree of mapping = net number of rounds (CCW=positive).

Lemma: degree on boundary = degree in interior.
Steps in Existence Proof

<table>
<thead>
<tr>
<th>Step</th>
<th>Proved for</th>
</tr>
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<tbody>
<tr>
<td>1. $n$ agent-labelings with permutation condition $\rightarrow$ Combined labeling with permutation condition</td>
<td>Any $n$</td>
</tr>
<tr>
<td>2. Permutation condition $\rightarrow$ Nonzero boundary degree</td>
<td>$n = 3$</td>
</tr>
<tr>
<td>3. Boundary degree $= \text{Interior degree}$</td>
<td>Any $n$ (?)</td>
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Step 1: $n$ labelings $\rightarrow$ 1 labeling

We need to assign owners to vertices s.t.:

- In each sub-simplex, each vertex belongs to a different owner.
- Friends are assigned to the same owner.

Does not work with the equilateral triangulation.
Step 1: $n$ labelings $\rightarrow$ 1 labeling

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**Lemma:** it works with *barycentric triangulation*
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Step 2: Permutation → Boundary degree

Permutation condition:

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Lemma: *When* $n=3$, if labeling satisfies *permutation condition* and *agent condition*, then labels on main vertices can be chosen such that: boundary-degree mod 3 $\neq 0$.  

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**Step 2: Permutation → Boundary degree**

**Permutation condition:**

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**Agent condition:**

$\text{deg} = 3k + 1$
Step 2: Permutation → Boundary degree

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Agent condition: Either:

(+) In each main-vertex \( i \), the label is \( i \), or:

(-) In each main-vertex \( i \), the label can be anything but \( i \).

2 of 9 cases shown below:

\[
\text{deg} = (3k) - 2/3 - 1/3 = 3k - 1 \quad \text{deg} = (3k - 1) - 1/3 + 1/3 = 3k - 1
\]
Step 3: Boundary degree = Interior degree

Definition:

Degree of labeling of an $n$-simplex in $R^{n-1}$
= sign of determinant of affine transformation to $Q$
= +1 if onto&no reflection, -1 if onto&one reflection, 0 if not onto.
Step 3: Boundary degree = Interior degree

Definition:

Orientation of an \((n-1)\)-simplex in \(R^{n-1}\)
= one of its two adjacent half-spaces.

Degree of labeling of an \((n-1)\)-simplex in \(R^{n-1}\)
= sign of determinant of any affine transformation to \(Q\) that preserves the orientation.

\[ \text{deg}(f_1) = +1 \]
\[ \text{deg}(f_2) = -1 \]
Step 3: Boundary degree = Interior degree

Lemma:
Degree of a labeling of an \( n \)-simplex in \( \mathbb{R}^{n-1} \),
\[ = \text{sum of degrees on each face oriented } \text{inwards}: \]
Step 3: Boundary degree = Interior degree

Lemma:
Degree of a labeling of an $n$-simplex in $\mathbb{R}^{n-1}$, $= \sum$ of degrees on each face oriented *inwards*:

$$\deg = 1$$
Step 3: Boundary degree = Interior degree

**Lemma:**
Degree of a labeling of an $n$-simplex in $R^{n-1}$, 
= sum of degrees on each face oriented *inwards*:

\[
\text{deg} = \begin{cases} 
1 & \text{for orientation inward} \\
-1 & \text{for orientation outward} 
\end{cases}
\]
Step 3: Boundary degree = Interior degree

Lemma:
Degree of a labeling of an $n$-simplex in $R^{n-1}$, $\deg = \sum$ of degrees on each face oriented *inwards*:
Step 3: Boundary degree = Interior degree

Lemma:
Degree of a labeling of an $n$-simplex in $\mathbb{R}^{n-1}$, equal to the sum of degrees on each face oriented inwards: 

- $\text{deg} = 1$
- $\text{deg} = -1 + 1$
- $\text{deg} = -1$
- $\text{deg} = 0$
Lemma:
Sum of degrees of simplices in triangulation
= sum of degrees on each boundary face,
– since the internal faces cancel out:
**Step 3: Boundary degree = Interior degree**

**Definition:** degree of triangulation labeling

= sum of degrees of each sub-simplex labeling.
Step 3: Boundary degree = Interior degree

Definition: degree of triangulation labeling
= sum of degrees of each sub-simplex labeling.

Lemma: interior degree = sum of degrees on faces
= sum of degrees on faces of boundary = boundary degree.
## Conclusion

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**Theorem:** for 3 agents with continuous valuations, an envy-free connected division exists for arbitrary mixed valuations.
Open question

Permutation condition for 4 or more agents:

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<tr>
<th>Pref:</th>
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<th>Right</th>
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Conjecture: If labeling satisfies

*permutation condition* and *agent condition*,
then boundary-degree mod $n$ <> 0.

*If conjecture is true, then connected envy-free division exists for arbitrary mixed valuations!*
Open question
Dividing Goods that are Bads
(Midrash Rabba, Genesis 33:1)
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