"DIVIDE THE LAND EQUALLY" (Ezekiel 47:14)

Envy-Free Matchings in Bipartite Graphs

and their Applications to Fair Division

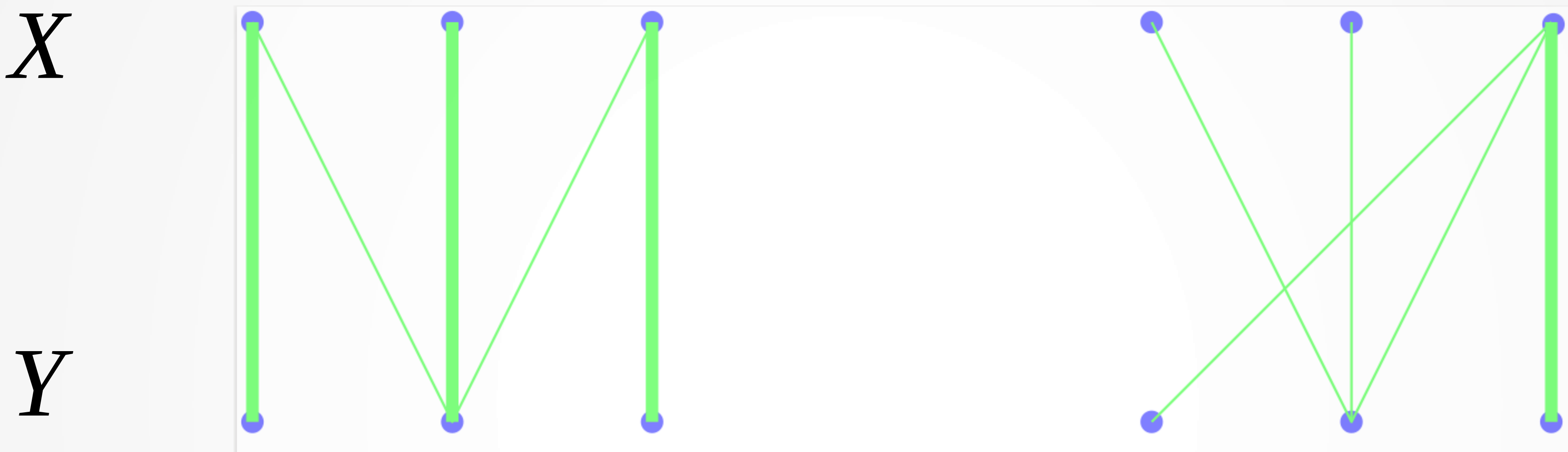
EREL SEGAL-HALEVI

JOINT WORK WITH

ELAD AIGNER-HOREV

ARIEL UNIVERSITY

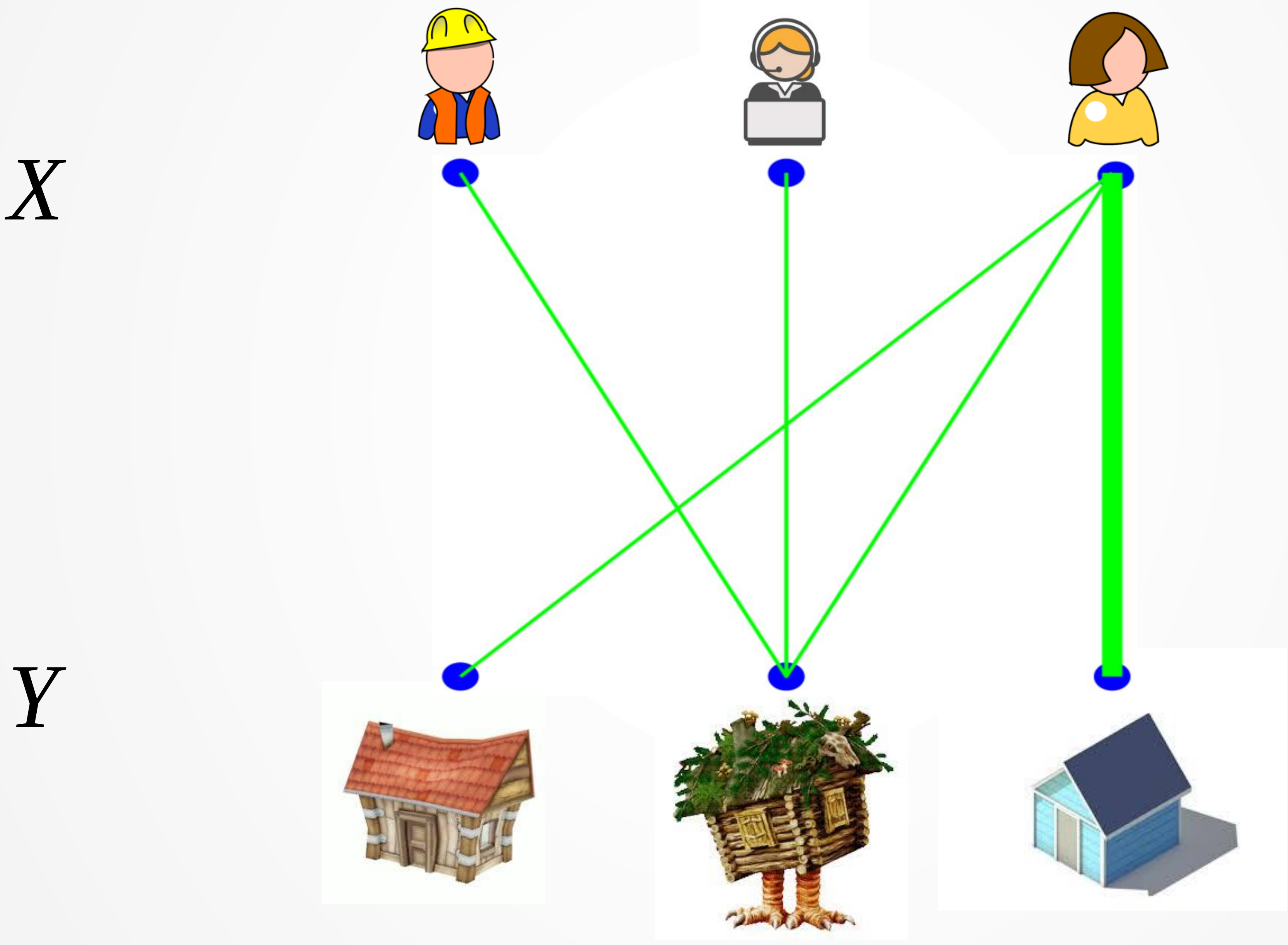
Perfect vs. Envy-Free Matching



X-saturating matching: *Every vertex of X is matched.*

Envy-free matching: *Every unmatched vertex of X is dis-connected from any matched vertex of Y.*

Envy-Free Matching: Metaphor



Envy-Free Matching: Existence

Question. Does an EFM always exist?

Envy-Free Matching: Existence

Question. Does an EFM always exist?

Answer. Yes – the empty-matching is EF.

Envy-Free Matching: Existence

Question. Does an EFM always exist?

Answer. Yes – the empty-matching is EF.

Question 2. Does a non-empty EFM always exist?

Envy-Free Matching: Existence

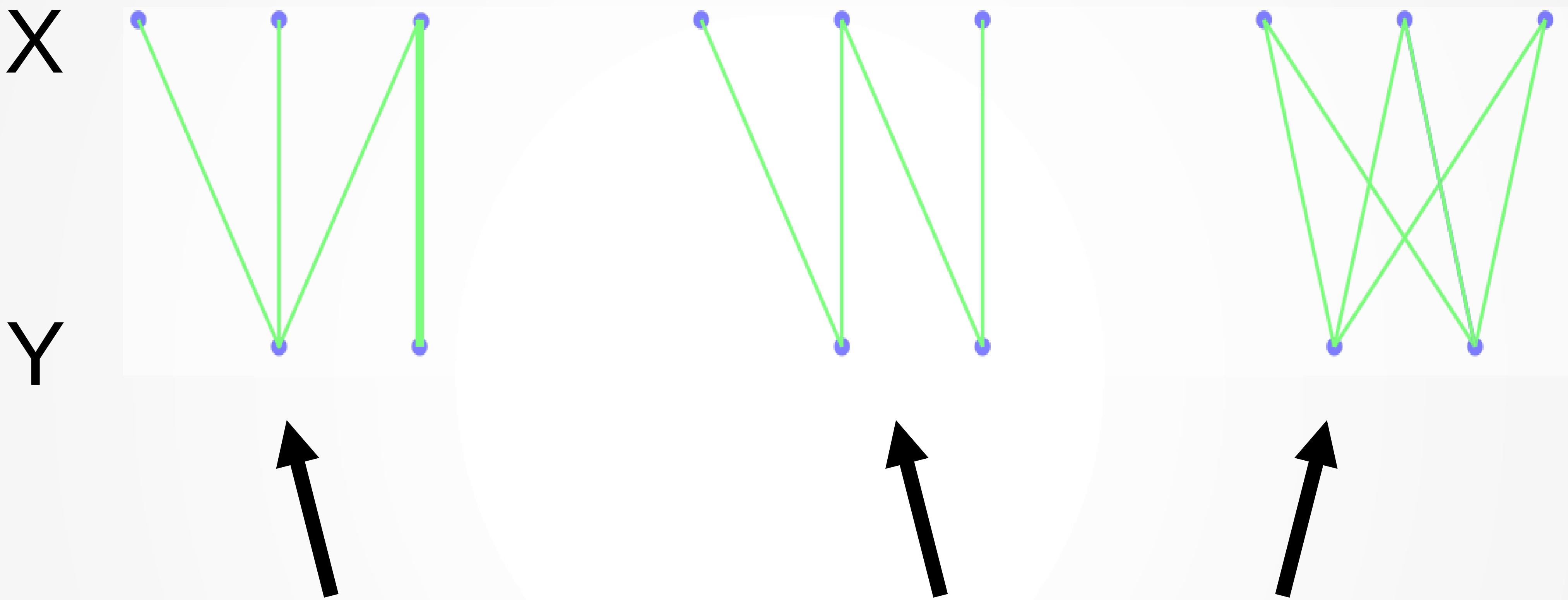
Question. Does an EFM always exist?

Answer. Yes – the empty-matching is EF.

Question 2. Does a non-empty EFM always exist?

Answer 2. No →

Non-empty vs. empty EFM



Non-empty EFM

Only empty EFM

Questions

1) Theory: What characterizes the graphs that admit a non-empty EFM?

2) Computation: How can we find an EFM of maximum size?

3) Application: What can we do with the unmatched vertices?

1. EFM and graph structure

Two extreme types of bipartite graphs:

- X**-saturated**: largest possible EFM.
- **Odd path**: only an empty EFM.

Theorem 1 (informal).

- Every G has a unique decomposition:
  G := X-saturated + “Odd-path-like”.
- Every EFM in G is contained in the X-saturated part.

1. EFM and graph structure

Definition. $G = (X \cup Y, E)$ is *odd-path-like* if, for some $k \geq 1$, there exist partitions

$$X = X_0 \sqcup X_1 \sqcup \cdots \sqcup X_k$$

$$Y = Y_1 \sqcup \cdots \sqcup Y_k$$

such that for all i ≥ 1

- X_i is perfectly matchable to Y_i;
- Every vertex in Y_i is adjacent to a vertex in X_{i-1}.

1. EFM and graph structure

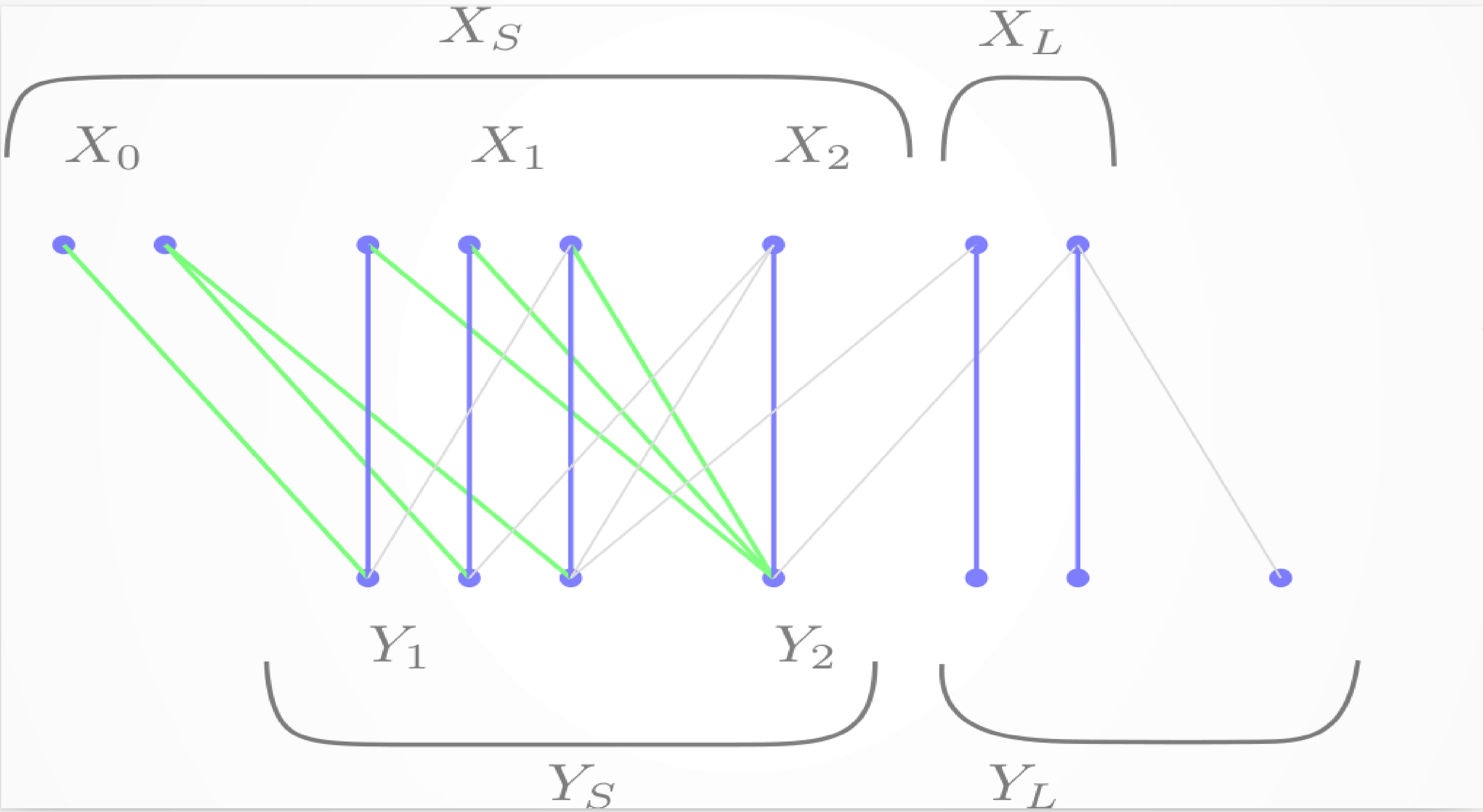
Theorem 1. Every bipartite $G = (X \cup Y, E)$ admits unique partitions

$$X = X_S \cup X_L \qquad Y = Y_S \cup Y_L$$

With the following properties:

a) No edges between X_S and Y_L;
b) $\mathrm{G}[X_S, Y_S]$ is odd-path-like;
c) $\mathrm{G}[X_L, Y_L]$ is X-saturated.
   Moreover:
d) Every X-sat. matching in $\mathrm{G}[X_L, Y_L]$ is EFM
e) Every EFM in G is contained in $\mathrm{G}[X_L, Y_L]$.

Theorem 1: Example



Theorem 1: Construction

- Take a maximum-size matching M.
- Let X_0 be the unmatched vertices in X.
- Construct a sequence of vertex subsets:

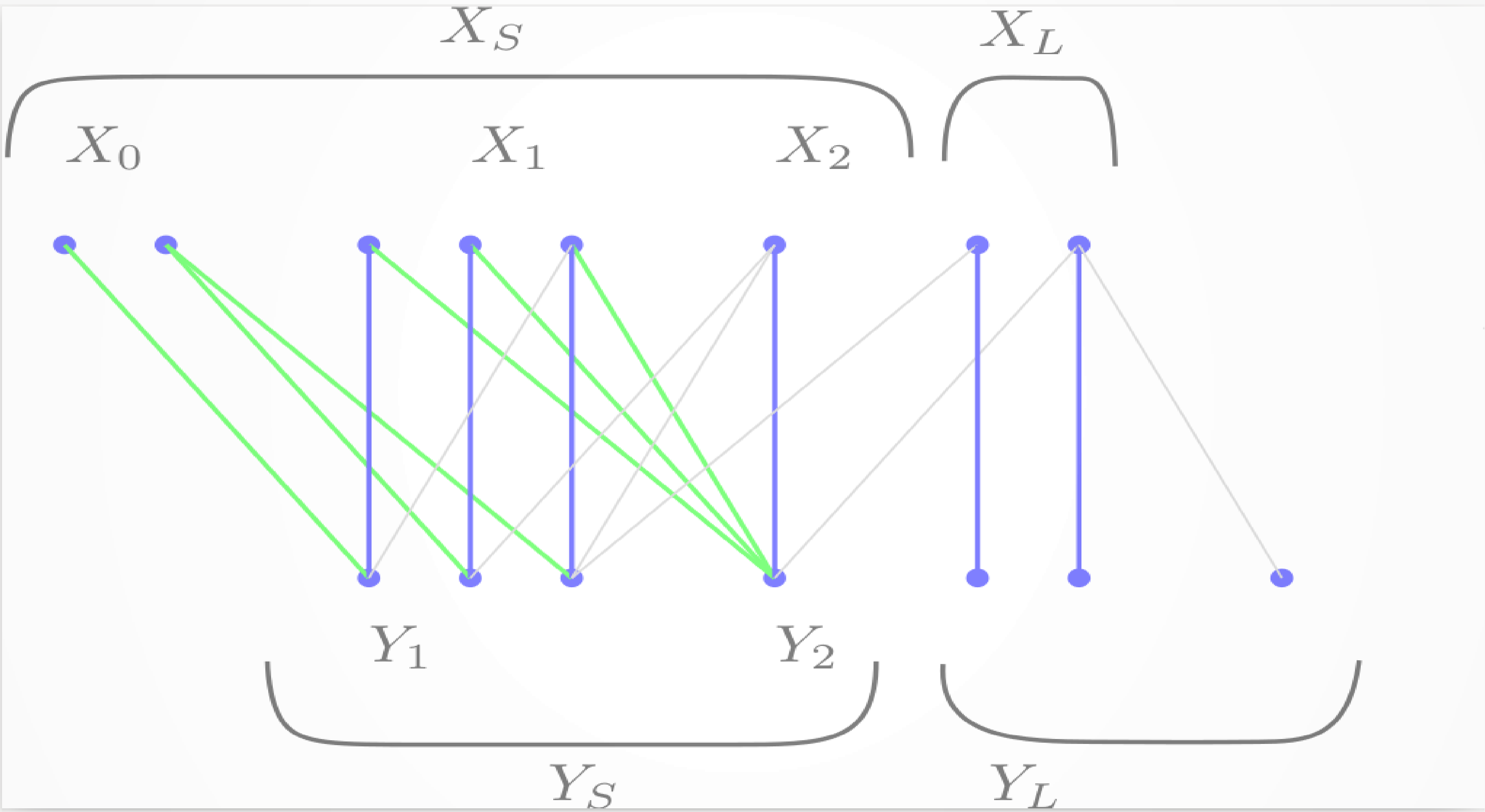
$$X_0 - Y_1 - X_1 - Y_2 - X_2 - \ldots -$$

where:

- $Y_i = \mathrm{N}_{G \setminus M}(X_{i-1}) \setminus \cup_{j<i} Y_j$;
- $X_i = \mathrm{N}_M(Y_i)$

- Let $X_S = \text{Union of } X_i$, $Y_S = \text{Union of } Y_i$,
  $X_L = X - X_S$, $Y_L = Y - Y_S$

Theorem 1: Construction



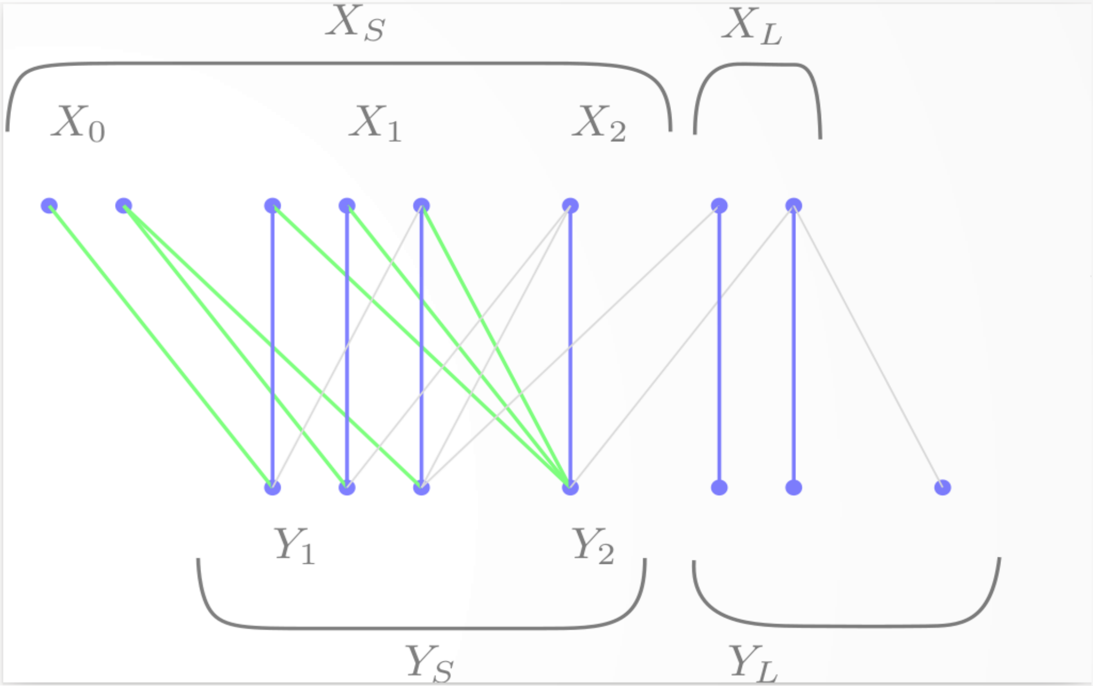
M = blue vertical lines

1 (proof): Decomposition

Decomposition:

X_S, Y_S = in sequence;

X_L, Y_L = the leftovers.



Properties:

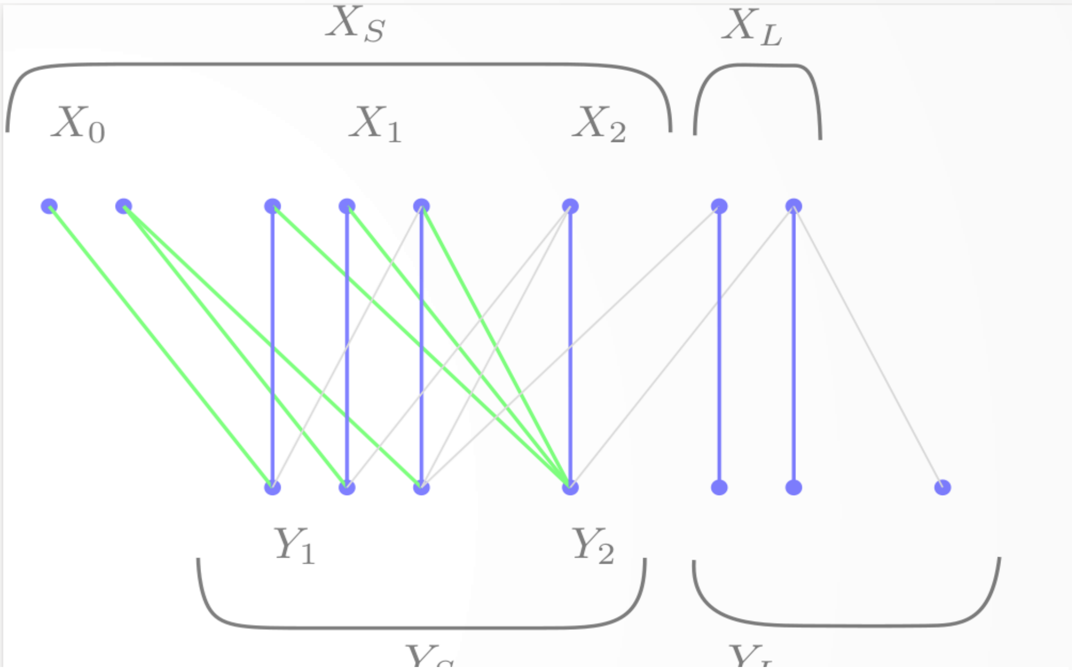
a) No edges between X_S and Y_L;

b) G[X_S, Y_S] is odd-path-like (construction ends at X side);

c) G[X_L, Y_L] is X-saturated (by edges of M).

1 (proof): Decomposition

Lemma. For any decomposition $G[X_S, Y_S] + G[X_L, Y_L]$ that satisfies properties (a),(b),(c):



(d) Every X-saturating matching in $G[X_L, Y_L]$ is envy-free in G.

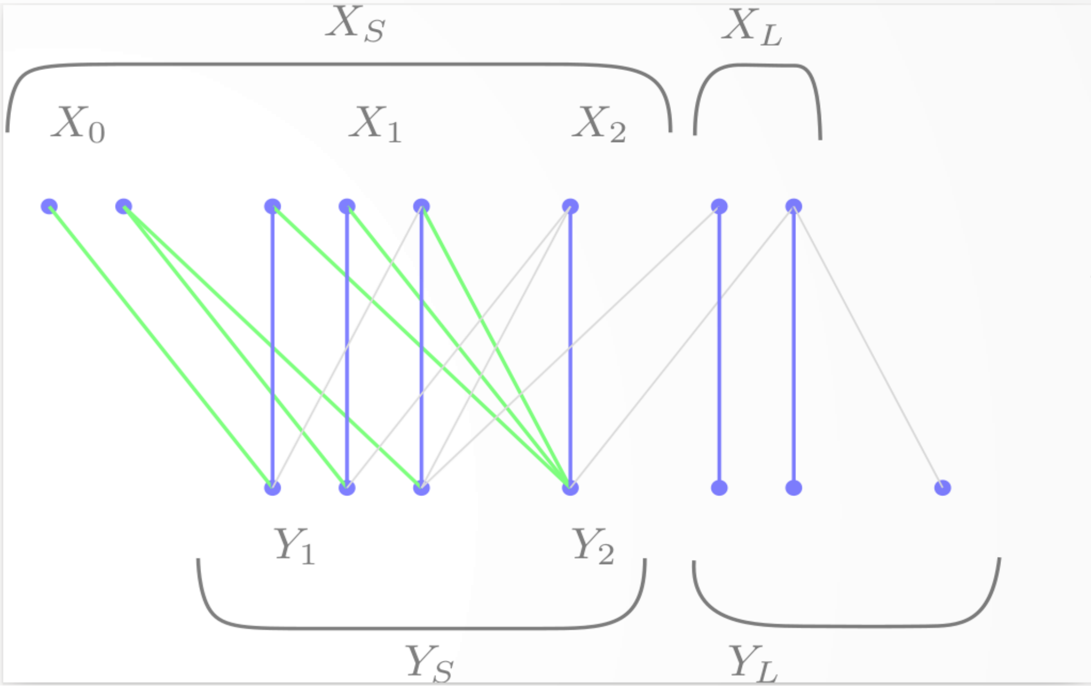
(e) Every envy-free matching in G is contained in $G[X_L, Y_L]$.

1 (proof): Decomposition

Proof of (d).

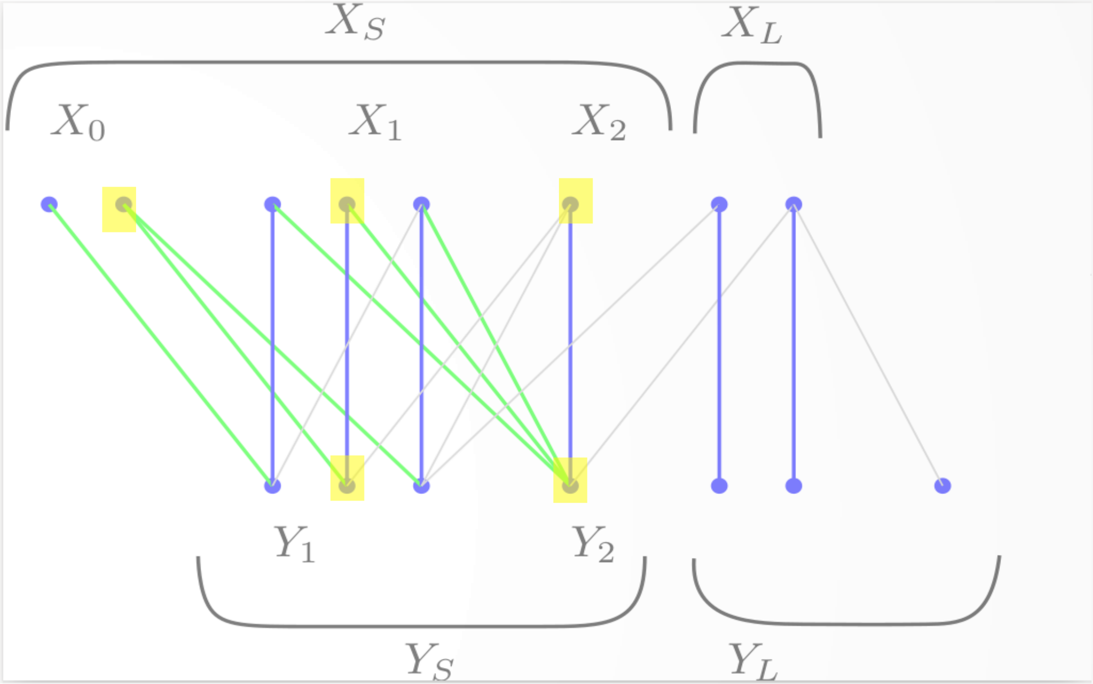
Given an X-saturating matching in G[X_L, Y_L]:

- Vertices of X_L do not envy since they are saturated.
- Vertices of X_S do not envy since by (a) they are not connected to Y_L.



→ The matching is envy-free in G.

1 (proof): Decomposition

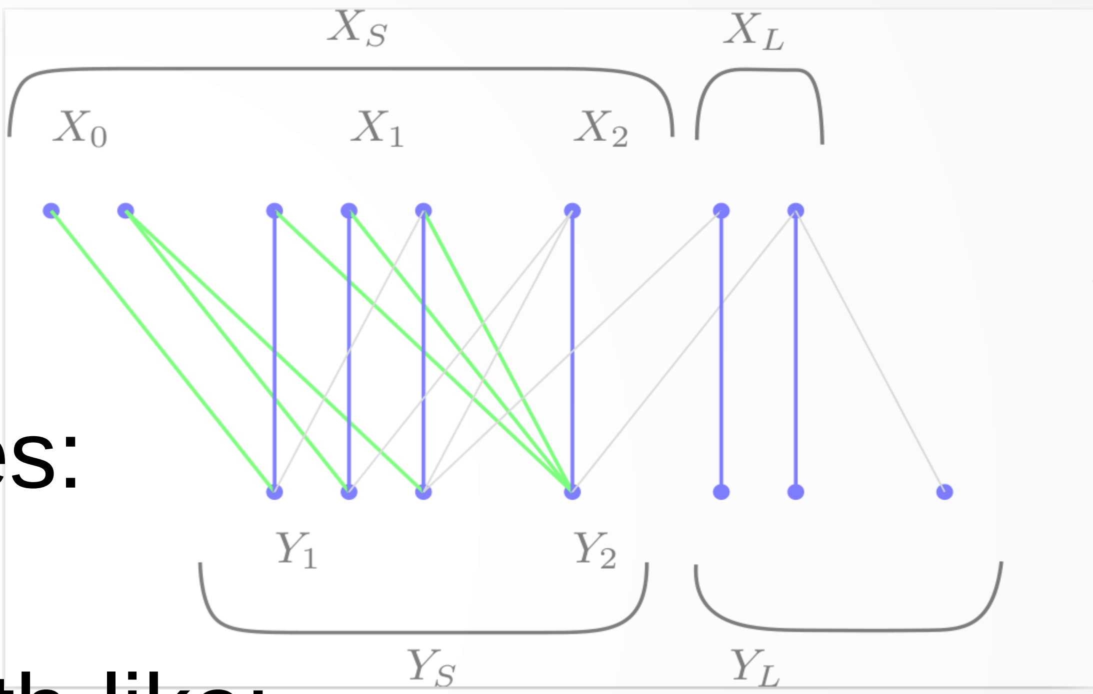


Proof of (e).

Given an envy-free matching W in G:

- i := smallest index s.t. a vertex in Y_i is matched by W.
- By (b), vertices of $Y_{\geq i}$ are perfectly matched. Their matches in $X_{\geq i}$ must be matched by W.
- At least one more vertex in X_{i-1} must be matched by W. → *Contradiction.*

1 (proof): Uniqueness



[Theorem 1] there is a *unique* decomposition $G[X_S, Y_S] + G[X_L, Y_L]$ satisfying the properties:

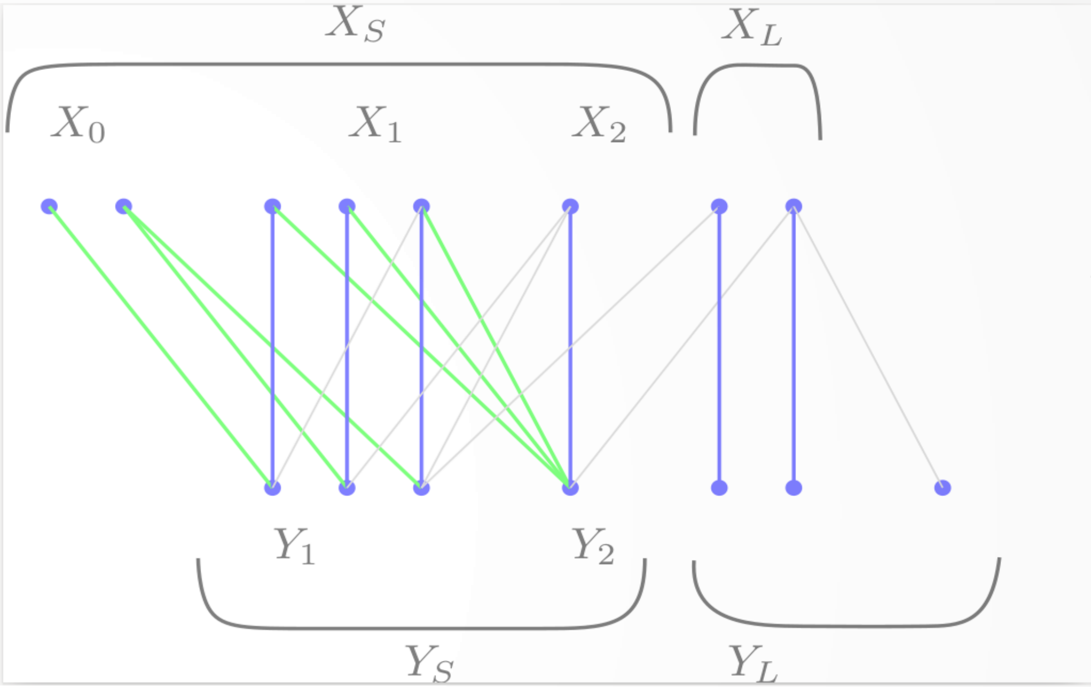
a) No edges X_S -- Y_L;
b) $G[X_S, Y_S]$ is odd-path-like;
c) $G[X_L, Y_L]$ is X-saturated.

Proof. Take any $G[X'_S, Y'_S] + G[X'_L, Y'_L]$.

- There is an EFM saturating X'_L, so $X'_L \subseteq X_L$.
- There is an EFM saturating X_L, so $X_L \subseteq X'_L$.

2. Algorithm for max-size EFM

1. Find a max-size matching M.
2. Construct the decomposition X_L, Y_L , X_S, Y_S.
3. Return $M[X_L, Y_L]$.

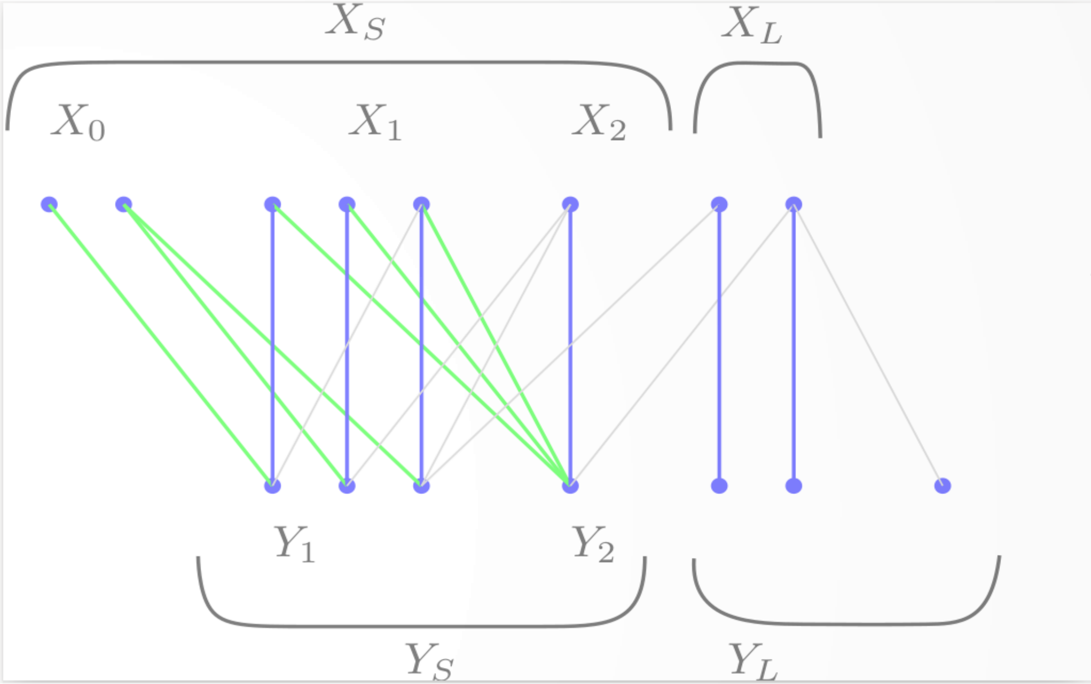


Correctness proof.

- By property (d), $M[X_L, Y_L]$ is an EFM.
- By property (e), there is no larger EFM.

2. Algorithm for max-size EFM

1. Find a max-size matching M.
2. Construct the decomposition $X_L, Y_L \;,\; X_S, Y_S$.
3. Return $M[X_L, Y_L]$.

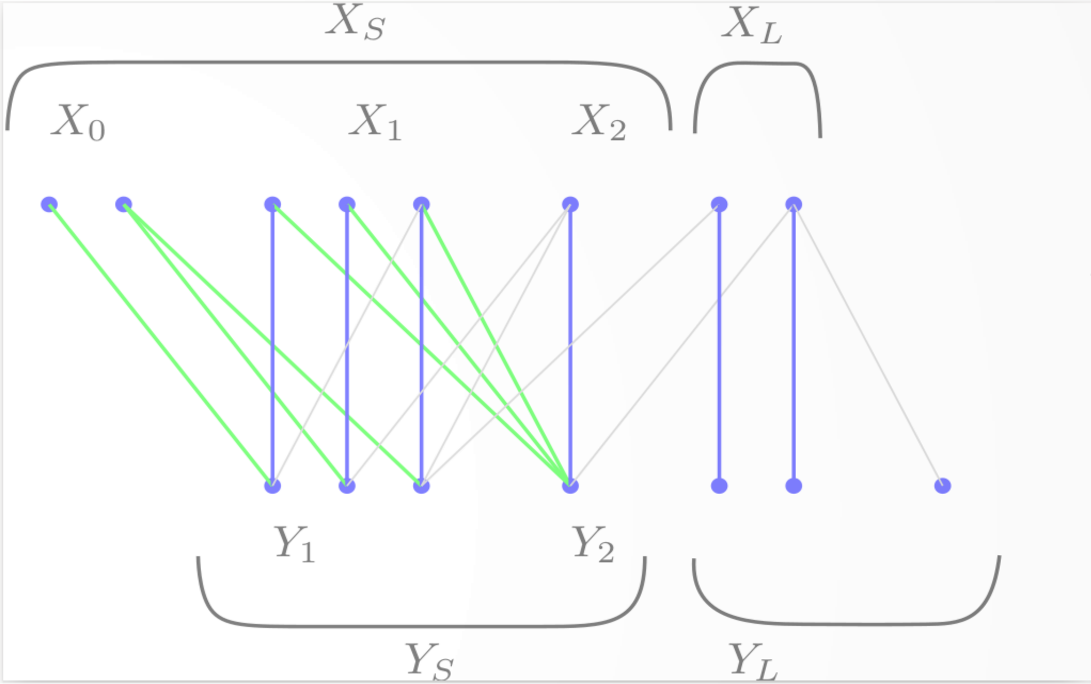


Extension.

- If each edge is endowed with a cost:
  - We can find a max-size min-cost EFM.

2. Algorithm for max-size EFM

1. Find a max-size matching M.
2. Construct the decomposition $X_L, Y_L \quad , \quad X_S, Y_S$.
3. Return $M[X_L, Y_L]$.



Corollary. $|N_G(X)| \geq |X| \geq 1$ → G has nonempty EFM.

Proof. It is sufficient to prove: $|X_L| \geq 1$.

- Case 1: $|X_0|=0$. Then $X_S = \emptyset$ so $X_L = X$ so $|X_L| \geq 1$.
- Case 2: $|X_0|>0$. Then $|X_S| > |Y_S| = |N_G(X_S)|$ →

$X_S \neq X$ → $|X_L| \geq 1$. ***

3. Applications for fair division

EFM can be used as a subroutine in various fair division problems:

(a) Fair *cake-cutting* – dividing a heterogeneous continuous resource;

(b) Fair *object allocation* – allocating discrete objects.

3 a. EFM in cake-cutting

INPUT:

- "Cake" – a heterogeneous divisible resource (e.g. land, time);
- Some n agents with different valuations (non-atomic measures) over the cake.

OUTPUT:

- Each agent gets a piece that he values as at least $1/n$ of the entire cake.

For 2 agents: *cut-and-choose.*

3 a. EFM in cake-cutting

ALGORITHM ("Lone Divider", Kuhn 1967):

1. Normalize cake value to n.
2. A (remaining) agent cuts n pieces worth 1.
3. Construct a bipartite graph $G[X,Y]$ with:
   * X = agents;
   * Y = pieces;
   * edge iff agent values piece at least 1.
4. Find in $G[X,Y]$ a maximum-size EFM.
5. Give each matched piece to its agent.
6. Update $n;$ if $n \geq 1$ go back to step 2.

3 a. EFM in cake-cutting

Proof of correctness.

4. $|N_G(X)| \geq |X| \geq 1$ → G has nonempty EFM.

5. Matched agents value their piece at ≥ 1.
   Unmatched agents value given pieces at <1.

6. The unmatched n-k agents
   value the remaining cake at > n-k.   ***

3 b. EFM in object-allocation

INPUT:

- Some discrete objects (e.g. house, car);
- Some n agents with different valuations (additive set functions) over the objects.

OUTPUT:

- Each agent gets a bundle worth for him at least his “1-out-of-n maximin-share” →

Maximin share

1-of-c maximin-share (MMS): the value an agent can get by partitioning the objects into *c* piles and getting the worst pile.

Example: for c=2, the 1-of-c MMS of i is 5:

3 b. EFM in object-allocation

INPUT: Discrete objects and n additive agents.
OUTPUT: 1-out-of-n MMS division.

- For 2 agents – *cut-and-choose*.
- For 3 or more agents – *may not exist* (Procaccia & Wang 2014).
- 1-out-of-**(n+1)** MMS division – open problem.
- 1-out-of-**(2n-2)** MMS division – next slide →

3 b. EFM in object-allocation

ALGORITHM:

1. Normalize 1-out-of-(2n-2) MMS to 1.
2. A remaining agent makes n bundles worth $\geq$1.
3. Construct a bipartite graph $G[X,Y]$ with:
   * X = agents;
   * Y = bundles;
   * edge iff agent values bundle at least 1.
4. Find in $G[X,Y]$ a maximum-size EFM.
5. Give each matched bundle to its agent.
6. Update n; if $n \geq 1$ go back to step 2.

3 b. EFM in object-allocation

Proof of correctness.

4. $|N_G(X)| \geq |X| \geq 1$ → G has nonempty EFM.

5. Matched agents value their bundle at ≥ 1.
Unmatched agents value given bundles at <1.

6. Technical lemma: Each of the unmatched n-k agents can divide the remaining objects into n-k bundles worth at least 1. ***

3 b. EFM in object-allocation

A similar algorithm can find an algorithm for:

- 2-out-of-($3n$-2) MMS allocation;
- (l-1)-out-of-(ln-2) MMS allocation, for any l;
- 2/3-fraction 1-out-of-n MMS allocation;
- An individual criterion for each agent.

Future Work

1. Envy-free one-to-many matchings:
    - A vertex x in X is „envious“ iff another vertex in X is matched to more vertices in Y that are adjacent to x.
2. Approximately-envy-free matchings:
    - A vertex x in X is „envious“ iff at least k of his neighbors in Y are matched.
3. 1-out-of-(n+1) MMS allocation ?

Acknowledgments

Zur Luria

Yuval Filmus

Thomas Klimpel

Thank you for coming ☺