A truthful Multi Item-Type Double-Auction Mechanism

Erel Segal-Halevi

with

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Yonatan Aumann
Intro: one item-type, one unit

Buyers:

Sellers:

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Multi Item Double Auction
Intro: one item-type, one unit

$k=5$ efficient deals

Buyers:

Sellers:

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Multi Item Double Auction
Intro: one item-type, one unit

Gain from trade:

Buyers:

Sellers:

$k=5$ efficient deals

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Multi Item Double Auction
Price Equilibrium

$k=5$ efficient deals
Price Equilibrium

✓ Maximum gain

$k=5$ efficient deals

Price

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Price Equilibrium

✓ Maximum gain
✓ Handles traders with many item-types if they are Gross-Substitutes (= no complementarities)

$k=5$ efficient deals

Price

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Price Equilibrium

✓ Maximum gain
✓ Handles traders with many item-types if they are \textit{Gross-Substitutes} (= no complementarities)

✗ Not truthful

$k=5$ efficient deals
Some related work

**Bayesian prior:**
- Double auction: Xu et al. [2010], Loertscher et al. [2014], Blumrosen and Dobzinski [2014], Colini-Baldeschi et al. [2016].

**Prior-independent:**
- Single-sided auction: Cole and Roughgarden [2014], Dhangwatnotai et al. [2015], Huang et al. [2015], Morgenstern and Roughgarden [2015], Devanur et al. 2011], Hsu et al. [2016].

**Prior-free:**
- Single-sided auction: Goldberg et al. [2001-2006], Devanur et al. [2015], Balcan et al. [2007-2008]
- Double auction: McAfee [1992] →
McAfee (1992) (simplified)

$\kappa=5$ efficient deals

Buyer price

Seller price

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Multi Item Double Auction
McAfee (1992) (simplified)

✓ Truthful

$k=5$ efficient deals

Buyer price

Seller price

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McAfee (1992) (simplified)

- Truthful
- Gain: \((1 - 1/k)\)

\(k=5\) efficient deals

Buyer price

Seller price

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Truthful

Gain: \((1 - 1/k)\) of maximum

Only single item-type, single-unit

\(k=5\) efficient deals

Buyer price

Seller price
McAfee (1992) (simplified)

✓ Truthful
✓ Gain: \( (1 - 1/k) \) of maximum
✗ Only single item-type, single-unit

Extensions:
Babaioff et al. [2004-2006],
Gonen et al. [2007],
Blumrosen & Dobzinsky [2014] - Single item-type, Gain \( \sim \frac{1}{48} \).
## Prior-Free Double-Auctions

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Prior-Free Double-Auctions

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Our current assumptions:

1. Buyers – at most $g$ item-types, gross-substitute.
   Sellers – 1 item-type, decreasing marginal gain.

2. Large market – for each item-type $x$, $k_x \rightarrow \infty$;
   at most $m$ units per seller;

3. Bounded variability – $k_{max} / k_{min} \leq c$


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MIDA: Multi Item Double-Auction

a. Random halving.

b. Equilibrium calculation.

c. Posted pricing.

d. Random serial dictatorship.
MIDA step a: Random Halving
MIDA step a: Random Halving

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MIDA step b: Equilibrium Calculation

Gross-substitute traders $\rightarrow$ price-equilibrium exists.
MIDA step c: Posted Pricing

Left

Right

$p_R$

$p_L$

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Multi Item Double Auction
MIDA step d: Random Dictatorship

In case of over-demand/supply – randomize.
MIDA step d: Random Dictatorship

- Order buyers randomly;
- Order sellers randomly;
- First buyer buys from first sellers and goes home.
- Seller goes home when marginal gain < 0.
MIDA step d: Random Dictatorship

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**Theorem:** If each seller sells one item-type and has decreasing-marginal-gains, then MIDA is truthful.
MIDA: Estimating the gain-from-trade
Four ways to lose gain-from-trade

Left: $p^L$

Right: $p^R$

$OPT$

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Multi Item Double Auction

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Four ways to lose gain-from-trade

Efficient sellers quitting: loss for buyers

Left

Right

$p^L$

$p^R$

$p^{OPT}$

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Multi Item Double Auction

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Four ways to lose gain-from-trade

Efficient sellers quitting: loss for buyers

Inefficient buyers competing: loss for other buyers

Left

Right

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Multi Item Double Auction
Four ways to lose gain-from-trade

Efficient sellers quitting: loss for buyers

Inefficient buyers competing: loss for other buyers

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Multi Item Double Auction
Four ways to lose gain-from-trade

Efficient sellers quitting: loss for buyers

Inefficient buyers competing: loss for other buyers

Efficient buyers quitting

Inefficient sellers competing

Left

Right

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Multi Item Double Auction
Four ways to lose gain  (left market)

For every item-type $x$, define:

- $B_{x^*}$ — buyers who want $x$ in $p^{OPT}$
- $B_{x^-}$ — buyers who want $x$ in $p^{OPT}$ but not in $p^R$
- $B_{x^+}$ — buyers who want $x$ in $p^R$ but not in $p^{OPT}$
- $S_{x^*}$ — sellers who offer $x$ in $p^{OPT}$
- $S_{x^-}$ — sellers who offer $x$ in $p^{OPT}$ but not in $p^R$
- $S_{x^+}$ — sellers who offer $x$ in $p^R$ but not in $p^{OPT}$

We lose $|B_{x^-}| + |S_{x^+}|$ random sellers and $|S_{x^-}| + |S_{x^+}|$ random buyers. So:

$$E[Loss_x] \leq \left( |B_{x^-}| + |B_{x^+}| + |S_{x^-}| + |S_{x^+}| \right) / |B_{x^*}|$$
Bounding the loss

\[ \mathbb{E}[\text{Loss}_x] \leq \frac{(|B_{x-}| + |B_{x+}| + |S_{x-}| + |S_{x+}|)}{k_x} \]

**Price-equilibrium equations:** for every \( x \):

- **Global population:** \( |B_{x^*}| = |S_{x^*}| = k_x \)
- **Right market** \((R = \text{the subset sampled to Right})\):
  \[
  |B_{x^*}^R| + |B_{x^+}^R| - |B_{x^-}^R| = |S_{x^*}^R| + |S_{x^+}^R| - |S_{x^-}^R|
  \]
Bounding the loss

\[ E[\text{Loss}_x] \leq \left( |B_{x-}| + |B_{x+}| + |S_{x-}| + |S_{x+}| \right) / k_x \]

Price-equilibrium equations: for every \( x \):
- Global population: \( |B_{x*}| = |S_{x*}| = k_x \)
- Right market: \( (R = \text{the subset sampled to Right}) \):
  \[ |B_{x*}^R| + |B_{x+}^R| - |B_{x-}^R| = |S_{x*}^R| + |S_{x+}^R| - |S_{x-}^R| \]

Concentration bounds: w.h.p:
\[
\left| \frac{|B_{x*}^R|}{2} - \frac{|B_{x*}|}{2} \right| < err_x
\]
\[
\left| \frac{|S_{x*}^R|}{2} - \frac{|S_{x*}|}{2} \right| < err_x
\]
\[
err_x = m \sqrt{k_x \ln k_x}
\]
Bounding the loss

$$E[\text{Loss}_x] \leq \left( |B_{x-}| + |B_{x+}| + |S_{x-}| + |S_{x+}| \right) / k_x$$

Price-equilibrium + Concentration bounds:
With high probability:

$$|B_{x-}^R - B_{x+}^R| < 2 \text{ err}_x$$
$$|S_{x-}^R - S_{x+}^R| < 2 \text{ err}_x$$
Bounding the loss

\[ E[\text{Loss}_x] \leq \left( |B_{x-}| + |B_{x+}| + |S_{x-}| + |S_{x+}| \right) / k_x \]

Price-equilibrium + Concentration bounds:
With high probability:

\[ \left| |B_{x-}^R| - |B_{x+}^R| \right| < 2 \text{ err}_x \]
\[ \left| |S_{x-}^R| - |S_{x+}^R| \right| < 2 \text{ err}_x \]

Let’s focus on the buyers.

- **We have** bounds on: \[ \left| |B_{x-}^R| - |B_{x+}^R| \right| \]
- **We need** bounds on: \[ |B_{x-}|, |B_{x+}| \]
Bounding the loss: step A

- **We have bounds:** \( \|B_{x-}^R - B_{x+}^R\| < 2 \text{ err}_x \)

\[
\|B_{1-}^R - B_{1+}^R\| < 2 \text{ err}_1 \\
\|B_{2-}^R - B_{2+}^R\| < 2 \text{ err}_2 \\
\vdots \\
\|B_{g-}^R - B_{g+}^R\| < 2 \text{ err}_g
\]

- **We derive bounds on:** \( B_{x-}^R \) , \( B_{x+}^R \)
Bounding the loss: step A

- We have bounds: \( |B_{x-}^R| - |B_{x+}^R| < 2 \text{ err}_x \)
- We derive bounds on: \( |B_{x-}^R|, |B_{x+}^R| \)

\[
\begin{align*}
&\text{p}^R \text{ item } 1 \\
&\text{p}^R \text{ item } 2 \\
&\text{p}^R \text{ item } 3 \\
&\text{p}^R \text{ item } 4 \\
&\text{p}^\text{OPT} \\
\end{align*}
\]
Bounding the loss: step A

- We have bounds: \(|B_x^- R| - |B_x^+ R|\) < 2 err_x
- We derive bounds on: \(|B_x^- R|, |B_x^+ R|\)

---

\(p^R\) item 1
\(p^R\) item 2
\(p^R\) item 3
\(p^R\) item 4
\(p^{OPT}\)

**Theorem:** The demand of gross-substitute agents moves only downwards (Segal-Halevi et al, 2016).
Bounding the loss: step A

- We have bounds: \[ |B_{x-}^R| - |B_{x+}^R| < 2 \text{ err}_x \]
- We derive bounds on: \[ |B_{x-}^R|, |B_{x+}^R| \]

Theorem: The demand of gross-substitute agents moves only downwards (Segal-Halevi et al, 2016).
Bounding the loss: step A

- We have bounds: \[ \|B_{x^-}^R - B_{x^+}^R\| < 2 \text{ err}_x \]
- We derive bounds on: \[ |B_{x^-}^R|, |B_{x^+}^R| \]

For every item \( x \) that became cheaper:

\[ B_{x-}^R \subseteq \bigcup_{y < x} B_{y+}^R \]

- \[ \|B_{1^-}^R - B_{1^+}^R\| < 2 \text{ err}_{\text{max}} \]
- \[ \|B_{2^-}^R - B_{2^+}^R\| < 2 \text{ err}_{\text{max}} \]
- ... \[ \|B_{g^-}^R - B_{g^+}^R\| < 2 \text{ err}_{\text{max}} \]

- \[ |B_{1^-}^R| = 0 \quad \rightarrow \quad |B_{1^+}^R| < 2 \text{ err}_{\text{max}} \]
- \[ |B_{2^-}^R| < 2 \text{ err}_{\text{max}} \quad \rightarrow \quad |B_{2^+}^R| < 4 \text{ err}_{\text{max}} \]
- ... \[ |B_{g^-}^R| < 2^g \text{ err}_{\text{max}} \quad \rightarrow \quad |B_{g^+}^R| < 2^g \text{ err}_{\text{max}} \]
Bounding the loss: step B

- We have a bound: $|B_{x-}^R|, |B_{x+}^R| < 2g \text{ err}_{\text{max}}$
- We need a bound on: $|B_{x-}|, |B_{x+}|$

When $T$ is a deterministic set – (like $B_{x*}$) – determined before randomization –

w.h.p: $||T^R| - |T|/2| < \sqrt{|T| \ln |T|}$

$B_{x-}$ and $B_{x+}$ are random sets - depend on price – determined after randomization!

Our solution: bound the UI dimension of $B_{x-}, B_{x+}$
UI Dimension of Random Sets

UI Dimension – property of a random-set.

If \( \text{UIDim}(T) \leq d \) then (Segal-Halevi et al, 2017):

w.h.p: \[ |T^R| - |T|/2 | < d \cdot \sqrt{|T| \ln |T|} \]

1. **Containment-Order Rule**: If the support of \( T \) is ordered by containment, then \( \text{UIDim}(T) \leq 1 \).

2. **Union Rule**:
\( \text{UIDim}(T_1 \cup T_2) \leq \text{UIDim}(T_1) + \text{UIDim}(T_2) \)

3. **Intersection Rule**: If \( |T_1| < t \) then:
\( \text{UIDim}(T_1 \cap T_2) \leq \log(t)*(\text{UIDim}(T_1) + \text{UIDim}(T_2)) \)
Bounding the loss: step B

- We have a bound: \( |B_{x-}^R|, |B_{x+}^R| < 2^g \text{ err}_{\text{max}} \)
- We derive a bound on: \( |B_{x-}|, |B_{x+}| \)

**Lemma:** For every item-type \( x \):

\[
B_{x-} = B_{x*} \cap \bigcap_{X \ni x} \left( \bigcup_{Y \not\ni x} B_{X < Y} \right) \implies \text{UIDim}(B_{x-}) \leq 2^{2g \ln k_{\text{max}}}
\]

Similarly:
\[
\text{UIDim}(B_{x+}) \leq 2^{2g \ln k_{\text{max}}}
\]

**Corollary:** When \( k_{\text{max}} \gg 2^{3g} \), w.h.p:

\[
|B_{x-}|, |B_{x+}| < 3 \times (2^g \text{ err}_{\text{max}})
\]
Bounding the loss: step C

- We have a bound: \(|B_{x-}|, |B_{x+}| < 3 \times 2^g \times \text{err}_{\text{max}}\)
- Similarly:
  \(|S_{x-}|, |S_{x+}| < 3 \times 2^g \times \text{err}_{\text{max}}\)

- Lost deals in item x: < \(12 \times (2^g \times \text{err}_{\text{max}})\)
- Lost gain in item x < \(12 \times (2^g \times \text{err}_{\text{max}}) / k_x\)
- Lost gain overall < \(12 \times (2^g \times \text{err}_{\text{max}}) / k_{\text{min}}\)
- Lost gain overall < Const \(\cdot o(k_{\text{max}}) / k_{\text{min}}\)

Theorem: Under large-market assumptions, gain-from-trade of MIDA approaches maximum.
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