Wet Squares in the Desert

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1 Problem Statement

In the desert ($\mathbb{R}^2$) there are n water-pools (points). There are $n-1$ people, each of whom should be given a land-plot. All land-plots should be pairwise-interior-disjoint squares, parallel to each other (as is common in urban planning). Moreover, it is important that each square has sufficient access to water. Formally:

Definition 1. *Given a fixed set of* pools *(points in $\mathbb{R}^2$):*
(a) a wet square *is a square touching at least two pools.*
(b) a wet square-collection *is a collection of parallel pairwise-interior-disjoint squares, each of which is wet.*

Conjecture 1. *For every set of n pools, there exists a wet square-collection of cardinality at least $n-1$.*

If we replace "squares" with "rectangles", then the Conjecture is obviously true. For example, we can order the pools lexicographically by their x,y coordinates, and put a rectangle between each two adjacent pools in that order. However, this technique does not work with squares. I am trying to *refute* the Conjecture by searching for a set of n pools, such that the largest wet-square-collection has cardinality at most $n-2$.

2 Refutation in given coordinate system

Suppose we add an additional requirement: all squares must be parallel to a given pre-specified coordinate system (not only parallel to each other). With this additional requirement, the Conjecture is false. Consider the arrangement of 6 pools illustrated in Figure 1:

- A northern pool $A(0,18)$ and a southern pool $A'(0,-18)$;
- A western pool $B(-9,-2.7)$ and an eastern pool $B'(9,2.7)$.
- Two central pools $C(-3,-0.9)$ and $C'(3,0.9)$.

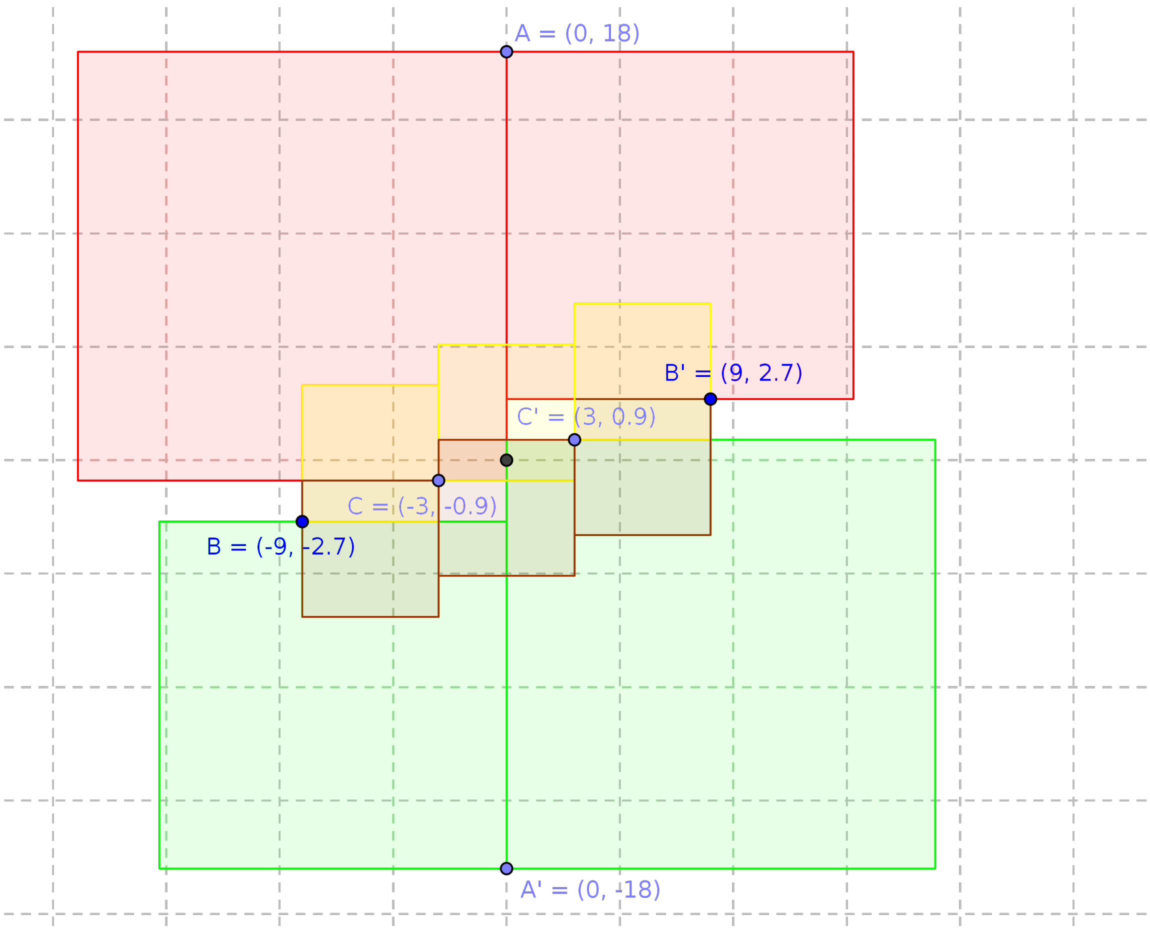

Figure 1: Impossibility result in a given coordinate system. Pools are indicated by blue points and letters. Note there is no pool in the origin (a black point). At most 4 disjoint axis-parallel squares touch more than one pool.
All figures were created with GeoGebra.

Claim 1. *With the pools of Figure 1, any wet-square-collection with axes-parallel squares contains at most 4 squares.*

Proof. Figure 1 illustrates the possible locations of wet-squares. Note that we can ignore squares that contain pools in their interior, since such squares can be shrunk such that the pools are on their boundary; such shrinking obviously does not interfere with other squares - it can only help. Hence, we consider:

- At most two interior-disjoint northern squares (red, touching pool A) and at most two interior-disjoint southern squares (green, touching pool A').
- At most one western square (touching pools B and C, can slide between the yellow and the brown) and at most one eastern square (touching pools B' and C', can slide similarly).
- At most one central square (touching pools C and C').

The key observation for the impossibility result is that *every central square intersects at least three northern/southern squares.* This observation is based on two facts:

Fact 1. *The northern and southern pools (A, A') lie horizontally between the two central pools (C, C').*

Fact 2. *The largest vertical space between a northern and a southern square (the vertical distance between B and B' = 5.4) is smaller than the smallest side-length of a central square (the horizontal distance between C and C' = 6).*

The combination of these two facts implies that, if a wet-square-collection contains a central square, it can contain at most one northern/southern square (e.g, the brown central square leaves room only for the northern AB', and the yellow central square leaves room only for the southern BA'). This implies that a wet-collection containing a central square can contain at most four squares (western, central, eastern, and northern/southern).

It remains to check the wet-square-collections that do not contain a central square. These can be of one of the following types (ordered in a decreasing order of the number of eastern/western squares):

- 2 eastern/western squares (BC and B'C') leave room to two northern/southern squares. E.g., two yellow squares and two green squares at their south, or a yellow BC and a brown B'C' and a green A'B and a red AB', or two brown squares and two red squares at their north.
- 1 eastern/western square leaves room to three northern/southern squares. E.g., yellow BC, two green squares and a red AB'.
- With no eastern/western squares, there is room for four northern/southern squares - two red and two green.

In any case, a wet collection contains at most 4 squares. □

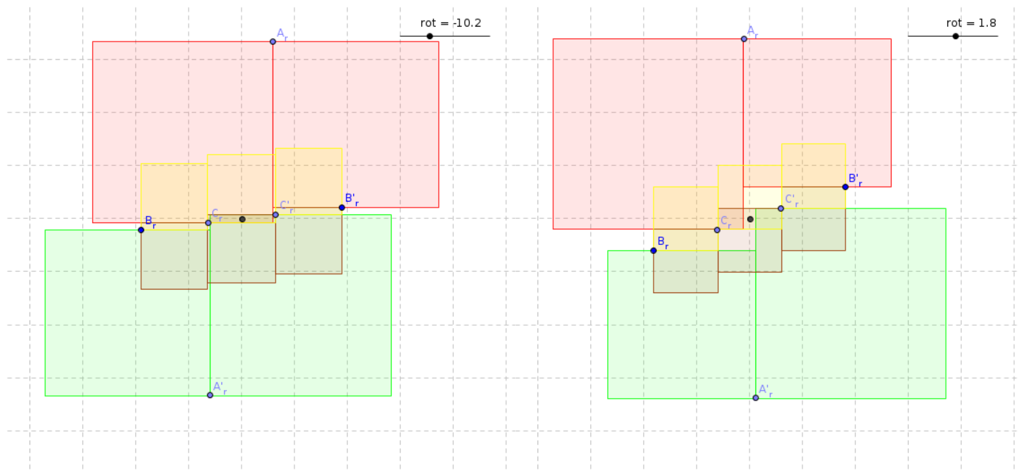

Figure 2: Impossibility result with a rotated arrangement. At most 4 disjoint axis-parallel squares touch more than one pool,

Figure 3: Duplicated arrangements in a row.

3 Rotated coordinate system

When the coordinate system is rotated, the results of the previous section hold as long as the two Facts are true. By rotating the arrangement of points, which is equivalent to rotating the coordinate system, the following bounds are revealed (see Figure 2):

- Fact 1 holds when the arrangement is rotated at most $10.2°$ clockwise;
- Fact 2 holds when the arrangement is rotated at most $1.8°$ counter-clockwise.

When the rotation angle is in $[-10.2°, 1.8°]$, both facts are true. This means that, with the original arrangement, if the squares must be parallel to each other and their rotation relative to the original axes must be in $[-1.8°, 10.2°]$, the size of any wet square-collection is still at most 4.

4 Duplicated arrangements

Take four arrangements of 6 pools, rotated in multiples of $12°$. For each such arrangement, there is a range of angles of size $12°$, such that any wet square-collection contains at most four squares with rotation in that range; the total

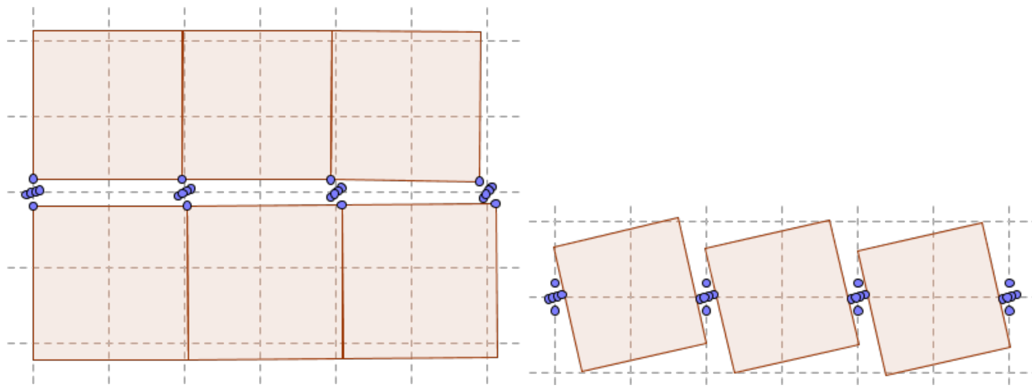

Figure 4: Why duplicated arrangements do not refute the conjecture.

range of angles covered is thus 48°. Every rotation angle of a square is equivalent to an angle in the range $[0°, 45°]$; hence, if the squares must be parallel, regardless of their rotation angle, then there is (at least) one arrangement in which the size of any wet collection is at most 4.

The problem with this idea is that there may be squares not only *within* arrangements but also *between* arrangements. For example, consider the 4 arrangements in Figure 3, containing 24 pools in total. Suppose that within one arrangement there are 4 wet-squares and within each of the other three arrangement there are 5 wet-squares. In addition to these 19 wet squares, it is possible to add 6 wet-squares as in Figure 4/Left, for a total of 25, which is more than the required 23.

Alternatively, suppose the four arrangements are not rotated. Then, the number of axes-parallel squares within each arrangement is only 4 and the total number of axes-parallel squares is $4 + 4 + 4 + 4 + 6 = 22$; however, we can now use rotated squares as in Figure 4/Right. Then, there are at least 5 wet-squares within each arrangement, and 3 additional wet-squares between arrangements, for a total of $5 + 5 + 5 + 5 + 3 = 23$.

5 Recursive arrangements

In all the discussion so far, it is not necessary to that the pools be points. All the results are still true when the pools are discs of sufficiently small radius. The advantage of this viewpoint is that it allows to create recursive arrangements.

Take an arrangement of 6 pools, shrink it to the size of a single pool, and put it instead of the northern pool in a rotated arrangement, as in Figure 5. The total number of pools is 11.

If the squares must be within the range of 12° covered by the large arrangement, then there are at most 4 pairwise-disjoint wet-squares touching one or more pools of the large arrangement.

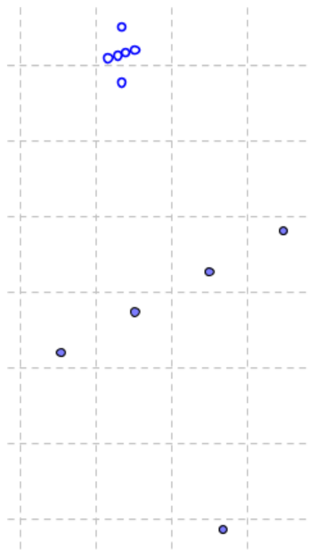

Figure 5: Recursive arrangements.

If the squares must be within the range of $12°$ covered by the small arrangement, then there are at most 4 pairwise-disjoint wet-squares touching two pools of the small arrangement.

Hence, if the squares must be within the range of $24°$ covered by the two arrangements, then at least one of the arrangements supports at most 4 wet-squares. If we could prove that the other arrangement supports at most 5 wet-squares, then we could conclude that the total number of squares is at most $4 + 5 = 9$. With two more recursive steps, we could cover the entire $45°$ range. The total number of pools is $6 + 5 + 5 + 5 = 21$ pools, and the size of any wet collection is at most $4 + 5 + 5 + 5 = 19$.

The problem here is that, we can not be sure that the size of any wet collection within an arrangement is at most 5.

6 Conclusion

The Conjecture has not been refuted - the question is still open.