"DIVIDE THE LAND EQUALLY" (Ezekiel 47:14)

Competitive Equilibrium For Almost All Incomes

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Inspired by: *Babaioff, Nisan and Talgam-Cohen (MATCHUP 2017): "Competitive Equilibria with Indivisible Goods & Generic Budgets."*

Fair Division of Indivisible Items

INPUT: m indivisible items.

n agents with strict monotone preferences on bundles:

GOAL: "Fair" allocation $X_1, \ldots, X_n$:

X_1  X_2

Fairness Criteria

Sylvain Bouveret & Michel Lemaître (2015). "Characterizing conflicts in fair division of indivisible goods using a scale of criteria". JAAMAS 30.

CE from Equal Incomes

CE from equal Incomes:=
allocation X & price-vector p such that:
1. For every agent i, $p(X_i) \leq 1$ (equal incomes)
2. Every agent i prefers X_i over all bundles with price at most 1. (CE)

- Always Pareto-efficient and envy-free;
- Nonexistent even for 2 agents, 1 item!
- Many previous works stop here.

CE from General **Incomes**

CE from **general** incomes $(t_1, \ldots, t_n)$:=
allocation X & price-vector p such that:

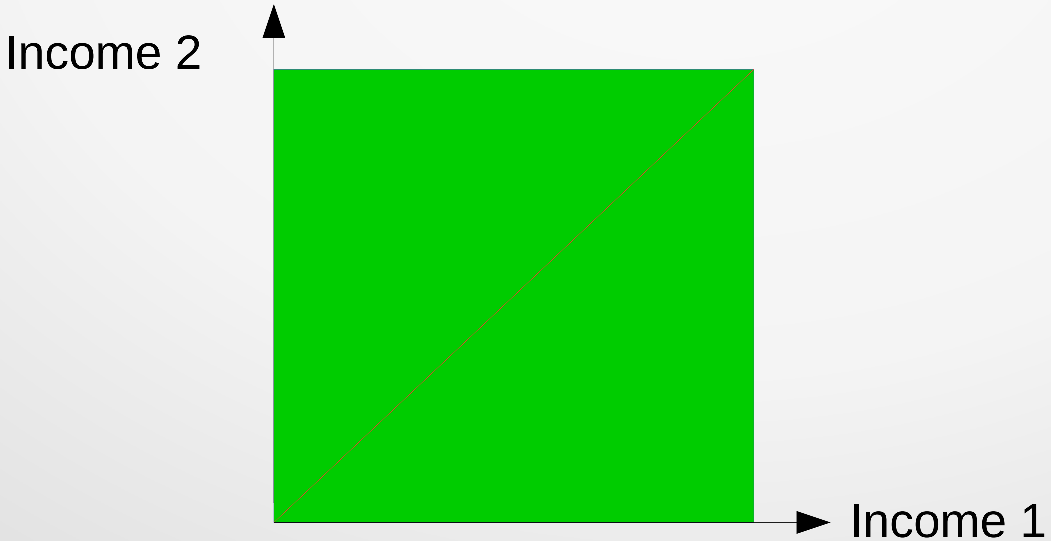
1. For every agent i, $p(X_i) \leq t_i$.
2. Every agent i prefers X_i over all bundles with price at most t_i. (CE)

- Still always Pareto-efficient;
- Satisfies *fairness with unequal entitlements;*
- ***With 1 item & 2 agents, CE exists iff $t_1 \neq t_2$!***

CE For Almost All Incomes

– so with 1 item and 2 agents, there exists a **CE F**or almost **A**ll **I**ncomes (= CEFAI) –

the subset of incomes without CE has measure 0: in the set of all incomes:



CE For Almost All Incomes – Questions

Q1: Does CEFAI always exist?

Previous answers:

Babaioff, Nisan, Talgam-Cohen (2017), "Competitive Equilibria with Indivisible Goods & Generic Budgets."

Items:	**1, 2, 3**	**4**	**5+**
2 agents:		Yes	
3 agents:	Yes	**???**	No
4+ agents:		**???**	

Q2: How to implement CE when it exists?

Picking Sequences

Picking sequence :=

- A protocol defined by m agent-names.
- Each agent in turn picks a single item.
- Simple, "elicitation free".
- Used e.g. for allocating cabinet ministries (Denmark, North Ireland, ...)

- Steven J. Brams and Todd R. Kaplan (2004): "Dividing the Indivisible". Journal of Theoretical Politics 16.
- Sylvain Bouveret and Jérôme Lang (IJCAI 2011): "A General Elicitation-free Protocol for Allocating Indivisible Goods".
- Thomas Kalinowski, Nina Narodytska, and Toby Walsh (IJCAJ 2013): "A Social Welfare Optimal Sequential Allocation Procedure".
- Haris Aziz, Paul Goldberg, and Toby Walsh (2017): "Equilibria in Sequential Allocation", ADT-17.

Picking Sequences with Prices

- **PIXEP :=** a **pick**ing-**se**quence with a **p**rice-tag attached to each position, e.g.:

Alice	Bob	Alice
4	2	1

- GOAL: prove that there exists a **subgame-perfect equilibrium** of the sequential game, such that the allocation & prices are a **CE.**

PIXEP example: 2 agents, 3 items

- Agents: A, B
- Incomes: a, b. W.l.o.g. $a>b>0$.
- PIXEP:

A	B	A
a-ε	b	ε

- Prices are decreasing – no agent can afford a picked item (necessary for CE).
- Analysis: Let z be Bob's worst item.
  - Suppose w.l.o.g. that for Alice: xz > yz.
  - Then the picks are **x, y, z** – it is a CE.

PIXEP: 3 or more agents, 3 items

- Agents: A, B, C, ...
- Incomes: $a > b > c > ...$
- If $a > b+c$ *(for sufficiently small $\varepsilon>0$):*

A	B	A
$a-c-\varepsilon$	b	$c+\varepsilon$

If $a < b+c$:

A	B	C
a	b	c

- Works for all incomes except when $a=b$ or $b=c$ or $a=b+c$ → **there is a CEFAI**.

PIXEP: 2 agents, 4 items

- Agents: A, B
- Incomes: $a > b$
- Protocol:

If a > 2 b:	**A**	**A**	**B**	**A**
	$a\text{-}b\text{-}2\varepsilon$	$b+\varepsilon$	b	ε
If a < 2 b:				
Alice may choose:	**A**	**B**	**A**	**B**
	$a\text{-}2\varepsilon$	$b\text{-}\varepsilon$	2ε	ε
Else, Bob may choose:	**B**	**A**	**A**	**A**
	b	$b\text{-}2\varepsilon$	$(a\text{-}b)/2+\varepsilon$	$(a\text{-}b)/2+\varepsilon$
Else, play:	**A**	**A**	**B**	**B**
	$a/2$	$a/2$	$b/2$	$b/2$

PIXEP: 3 agents, 4 items

- Agents: A, B, C
- Incomes: $a > b > c$
- Protocol →

(1) If $a > 2b + c$ then

A	A	B	A
$a-b-c^{--}$	b^{+}	b	c^{+}

(2) If $2b + c > a > 2b$ then

A	A	B	C
$a-b^{-}$	b^{+}	b	c

(3) If $2b > a > b+c$ & $a+c > 2b$ then

A	B	A	C
b^{+}	b	$a-b^{-}$	c

(4) If $2b > a > b+c$ and $2b > a+c$ (implies $b > 2c, a > 3c$) then:

Alice may choose:

A	B	A	B
$a-c^{--}$	$b-c^{-}$	c^{++}	c^{+}

Else, Bob may choose:

B	A	A	A
b	$a-2p^{--}$	p^{+}	p^{+}

where $p := \max\left(c,\ (a-b)/2\right)$

Else:

A	A	B	B
$a/2$	$a/2$	$b/2$	$b/2$

(5) If $b + c > a > 2c$ and $2c > b$ then play:

Alice may choose:

A	B	C	B
a	b^{-}	c	0^{+}

Else:

B	A	A	C
b	$a-c^{-}$	c^{+}	c

(6) If $b + c > a > 2c$ and $b > 2c$ then play:

Bob may choose:

A	B	A	B
$a-c^{--}$	$b-c^{-}$	c^{++}	c^{+}

Else, Alice may choose:

A	B	B	C
a	$b/2$	$b/2$	c

Else:

B	A	A	C
b	$a-c^{-}$	c^{+}	c

(7) If $2c > a$ then play the sequential game below:

Alice may choose:

A	B	C	B
a	b^{-}	c	0^{+}

Else:

B	A	C	A
b	c^{+}	c	$a-c^{-}$

IMPOSSIBILITY: 4 agents, 4 items

- Agents: A, B, C, D

- PREFERENCES:
  - Alice: xy > w > xz > yz > x > y > z
  - Bob: w > z > x > y
  - Carl: x > y > w > z
  - Dana: arbitrary

- INCOMES SUBSPACE:

  $2b > 2c > b+d > a > c+d > 2d > b > c > d$

- *– positive measure, no CE!*

IMPOSSIBILITY: 2 agents, 5 items

[Based on Babaioff, Nisan, Talgam-Cohen (2017)]

- Agents: A, B
- Preferences:
  - Alice: quartets > vwx, vwy, vwz > vw > xyz > vxy, vxz, vyz, wxy, wxz, wyz > pairs-except-vw > singletons
  - Bob: quartets > triplets-except-xyz > vx, vy, vz, wx, wy, wz > xyz > vw > v > w > xy, xz, yz > x, y, z
- INCOMES SUBSPACE: $a > b > 3a/4$
- *– positive measure, no CE!*

Conclusion

Complete characterization of CEFAI existence for **general monotone** prefs:

Items:	**1, 2, 3**	**4**	**5+**
2 agents:		Yes	
3 agents:	Yes	**Yes!**	No
4+ agents:		**No!**	

Next interesting questions

What happens when agents have **additive** valuations?

- 4 agents: No! (our example is additive).
- 3 agents: ??? *(my guess: No).*
- 2 agents: ??? *(my guess: Yes).*

CE fairness properties

Definition: Given a preference-relation $>_i$, a bundle X and two integers $l \le d$:

$$\begin{bmatrix} l \\ d \end{bmatrix} X \quad := \quad \max_{Y \in \textsc{Partitions}(X,d)}^{>_i} \quad \min_{Z \in \textsc{Unions}(Y,l)}^{>_i} \quad Z$$

Proposition: In any CE, for any agent i with preference $>_i$, any group of agents J and any two integers $l \le d$:

$$t_i \ge \frac{l}{d} \sum_{j \in J} t_j \quad \Longrightarrow \quad X_i \ge_i \begin{bmatrix} l \\ d \end{bmatrix} \bigcup_{j \in J} X_j$$

CE fairness properties

Interpretation: t_i is the *entitlement* of i.

Special case: with equal entitlements:

- envy free (take $l=d=1$, $t_i=t_j$).
- maximin share (take $l=1$, $d=n$, $t_1=...=t_n$).