

"וְצַחֲכֵתֶם אֹדֶתָה אִישׁ כְּאַחֲיוֹ" (יחזקאל' מז' 14)

Fairly Dividing a Cake after Some Parts were Burnt in the Oven

Erel Segal-Halevi

Fair Division — Definition

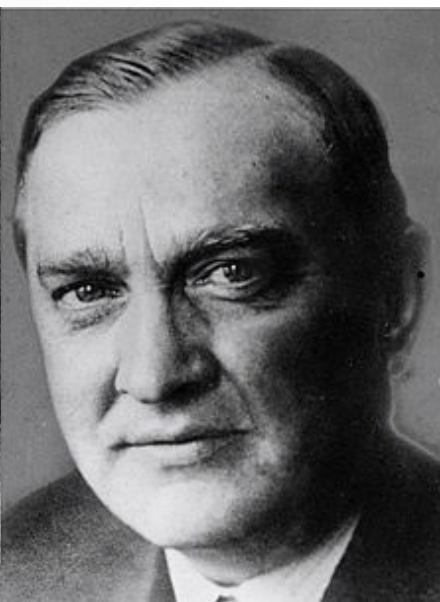
Dividing a **heterogeneous** resource to agents with **different** preferences such that everyone's share is "fair" **by their preferences.**

Fair Division — Then

Dividing a heterogeneous resource to agents with different preferences such that everyone's share is "fair" by their preferences.



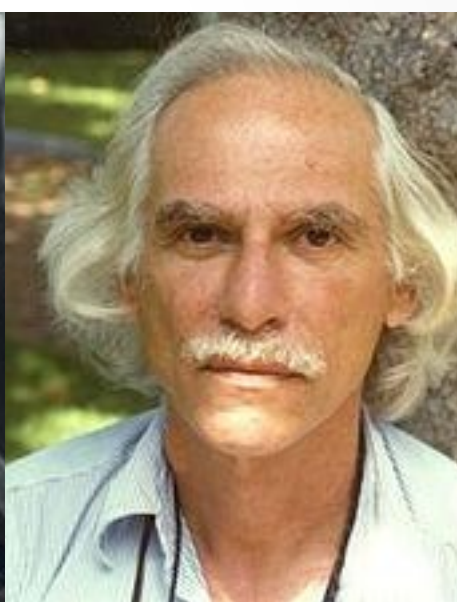
Steinhaus



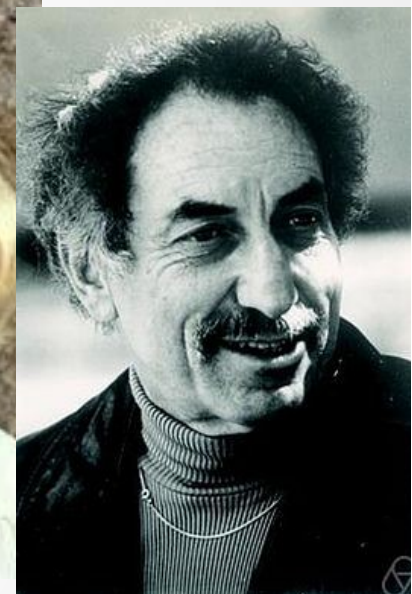
Banach



Knaster



Dubins



Spanier

Fair Division — Today

Dividing a heterogeneous resource to agents with different preferences such that everyone's share is "fair" by their preferences.

<http://fairoutcomes.com>

Fair Outcomes, Inc.

Game-Theoretic Solutions for Disputes and Negotiations
System Design - System Administration - Consultative and Online Services

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Fair Outcomes, Inc.

Fair Outcomes, Inc. provides parties involved in disputes or difficult negotiations with access to newly developed proprietary systems that allow fair and equitable outcomes to be achieved with remarkable efficiency. Each of these systems is grounded in mathematical theories of fair division and of games.

Our founders and staff include game theorists, computer scientists, and practicing attorneys with extensive experience in designing, administering, utilizing, and providing consulting and online services with respect to such systems.

Further information about our company and our services may be obtained by using the contact information appearing on this page. Additional information about four of our systems, each of which can be accessed and used online (and examined and tested free of charge), can be obtained by clicking on the links appearing below:



<http://spliddit.org>

PROVABLY FAIR SOLUTIONS.

Spliddit offers quick, free solutions to everyday fair division problems, using methods that provide indisputable fairness guarantees and build on decades of research in economics, mathematics, and computer science.

<https://math.hmc.edu/~su/fairdivision>

Francis Su's Fair Division Page

Click on [The Fair Division Calculator](#) which has recently been updated! (version 3.01, 4/12/00)

the
Fair Division
Calculator
v.3.0

A java applet for interactive decision making to find envy-free divisions of goods, burdens, or rent.



Share Rent

Moving into a new apartment with roommates? Create harmony by fairly assigning rooms and sharing the rent.

START >



Split Fare

Fairly split taxi fare, or the cost of an Uber or Lyft ride, when sharing a ride with friends.

START >



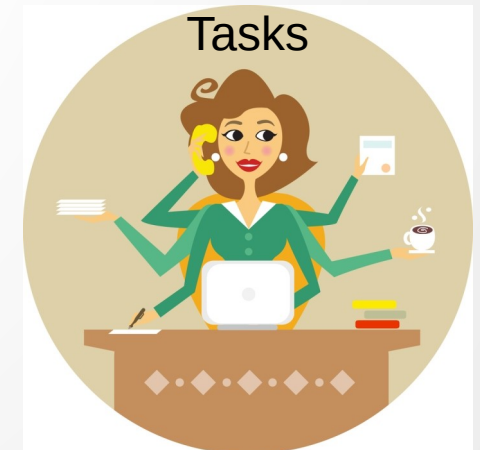
Assign Credit

Determine the contribution of each individual to a school project, academic paper, or business endeavor.

START >

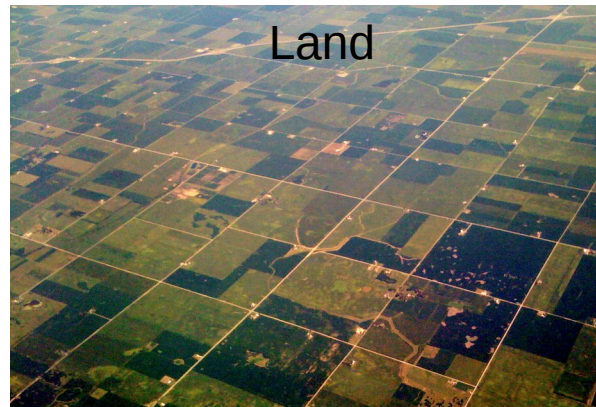
Fair Division — Examples

Dividing a heterogeneous resource to agents with different preferences such that everyone's share is "fair" by their preferences.



Fair Division — Examples

Dividing a **heterogeneous** resource to agents with **different** preferences such that everyone's share is "fair" by their preferences.



Continuous Resource

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Cake = Interval $[0,1]$.

n agents. Value-densities

$$v_i: \text{Cake} \rightarrow \mathbf{R}$$

Value = integral:

$$V_i(X_i) = \int_{X_i} v_i(x) dx$$

Continuous Resource

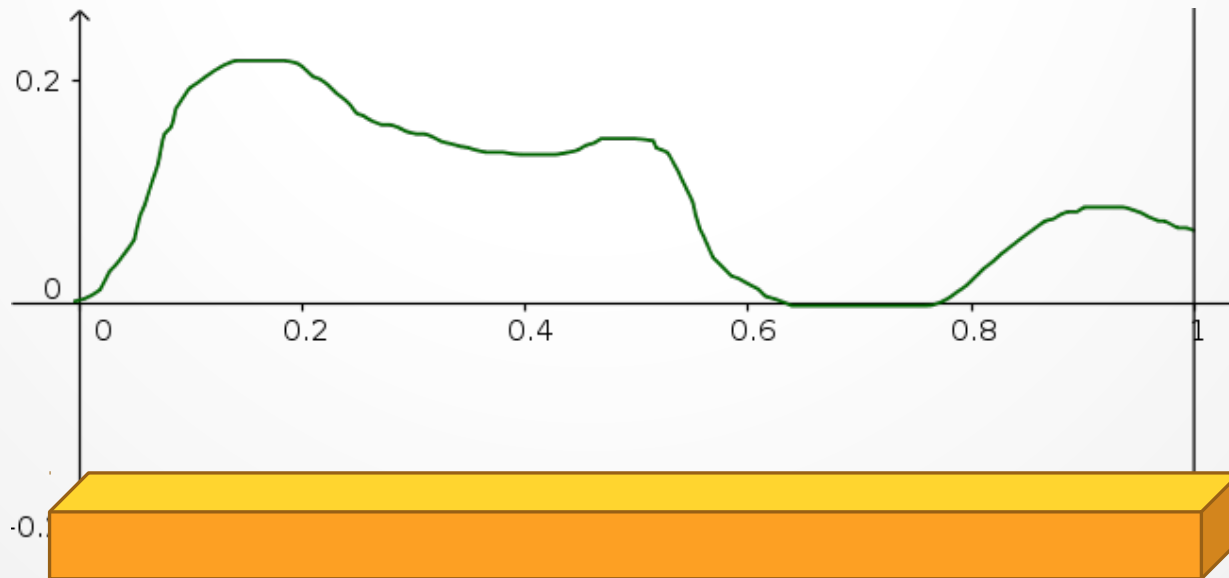
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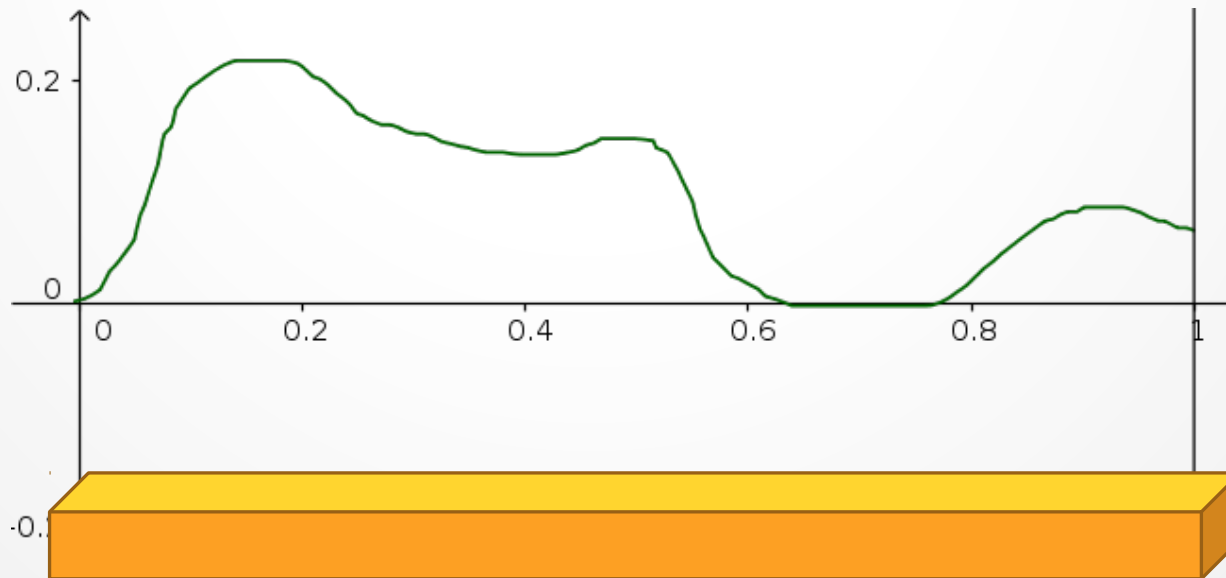
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For all i : X_i is **connected**.



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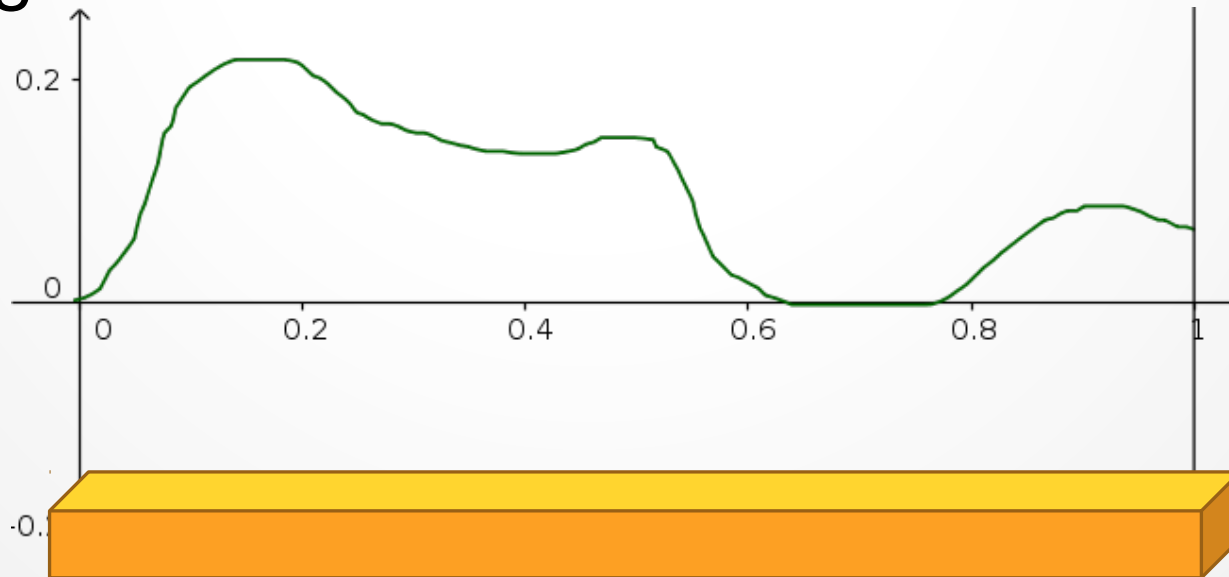
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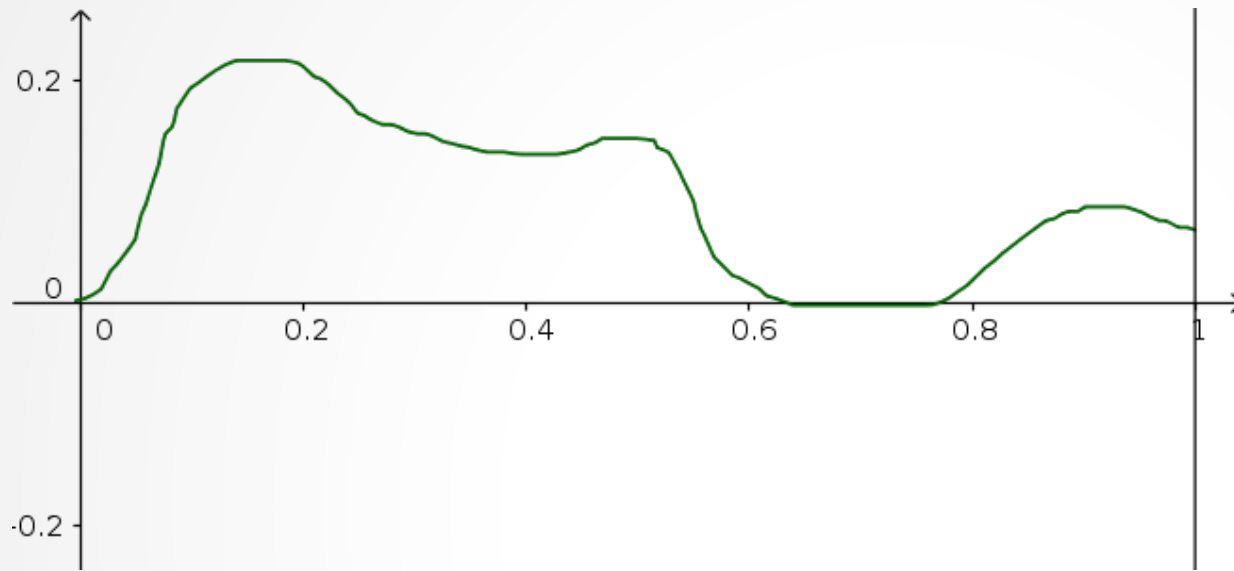
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Easy for 2 agents. **Difficult for 3 or more.**



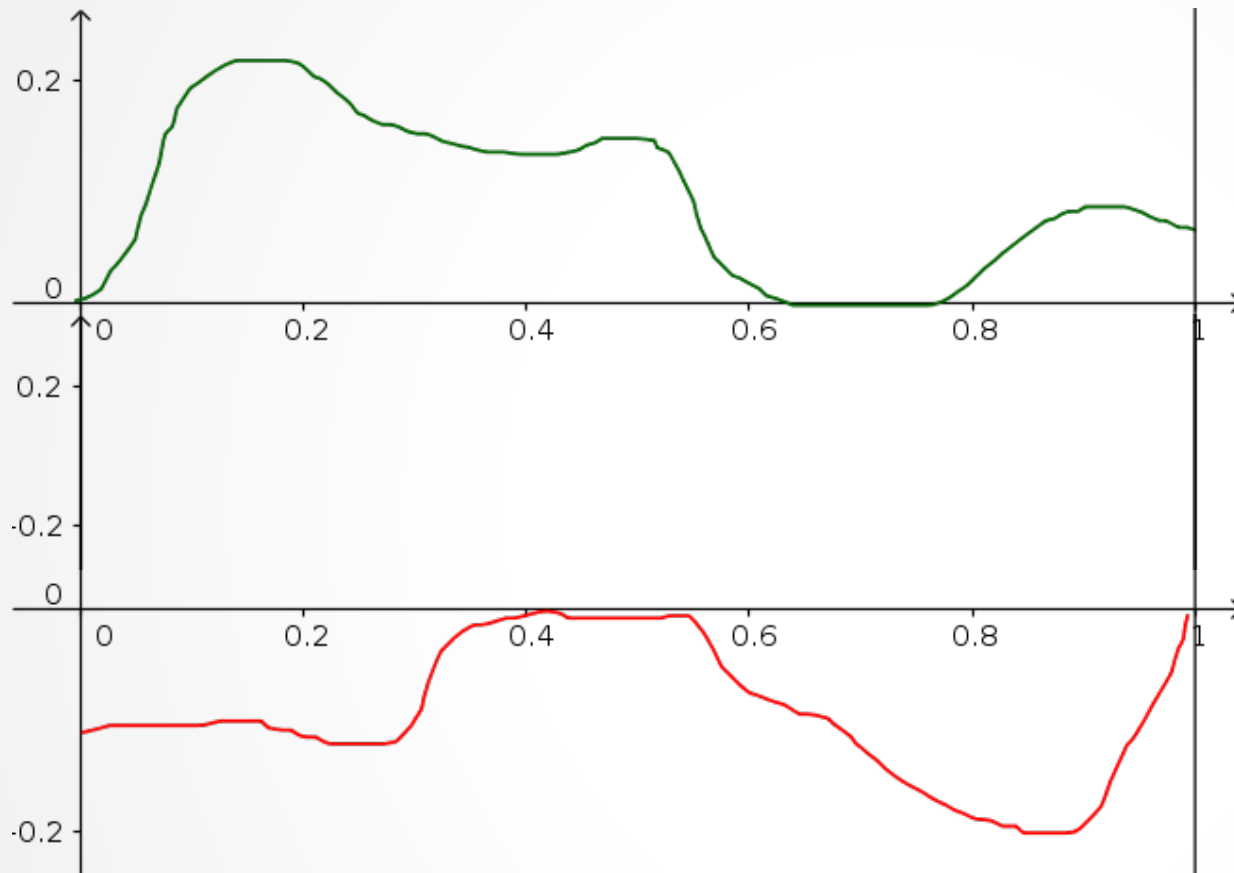
Valuation types

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All positive -
solved by
Stromquist (1980),
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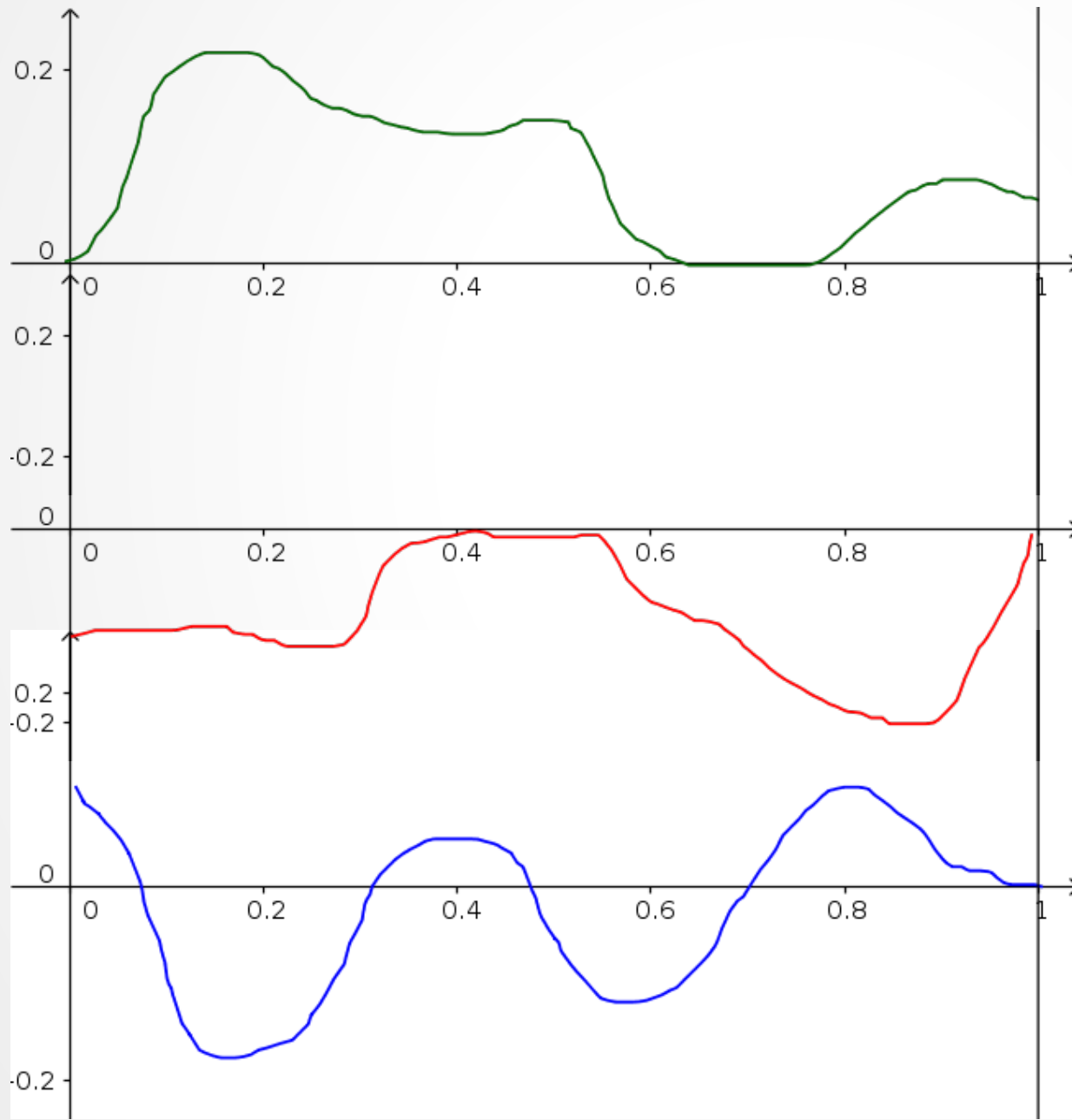
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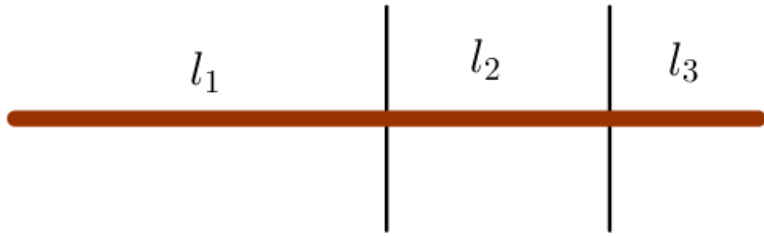
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General -
this work.

Simplex of Partitions – Definition

(based on Stromquist 1980)



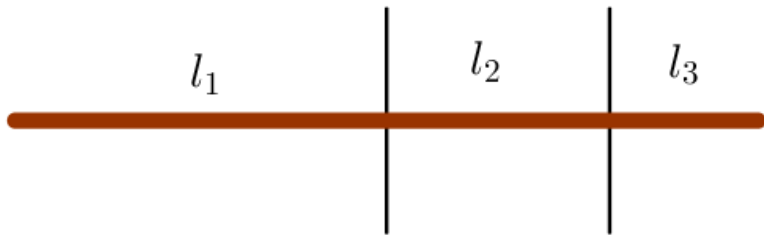
Partition for 3 agents:

$$(l_1, l_2, l_3)$$

$$l_1 + l_2 + l_3 = 1$$

Simplex of Partitions – Definition

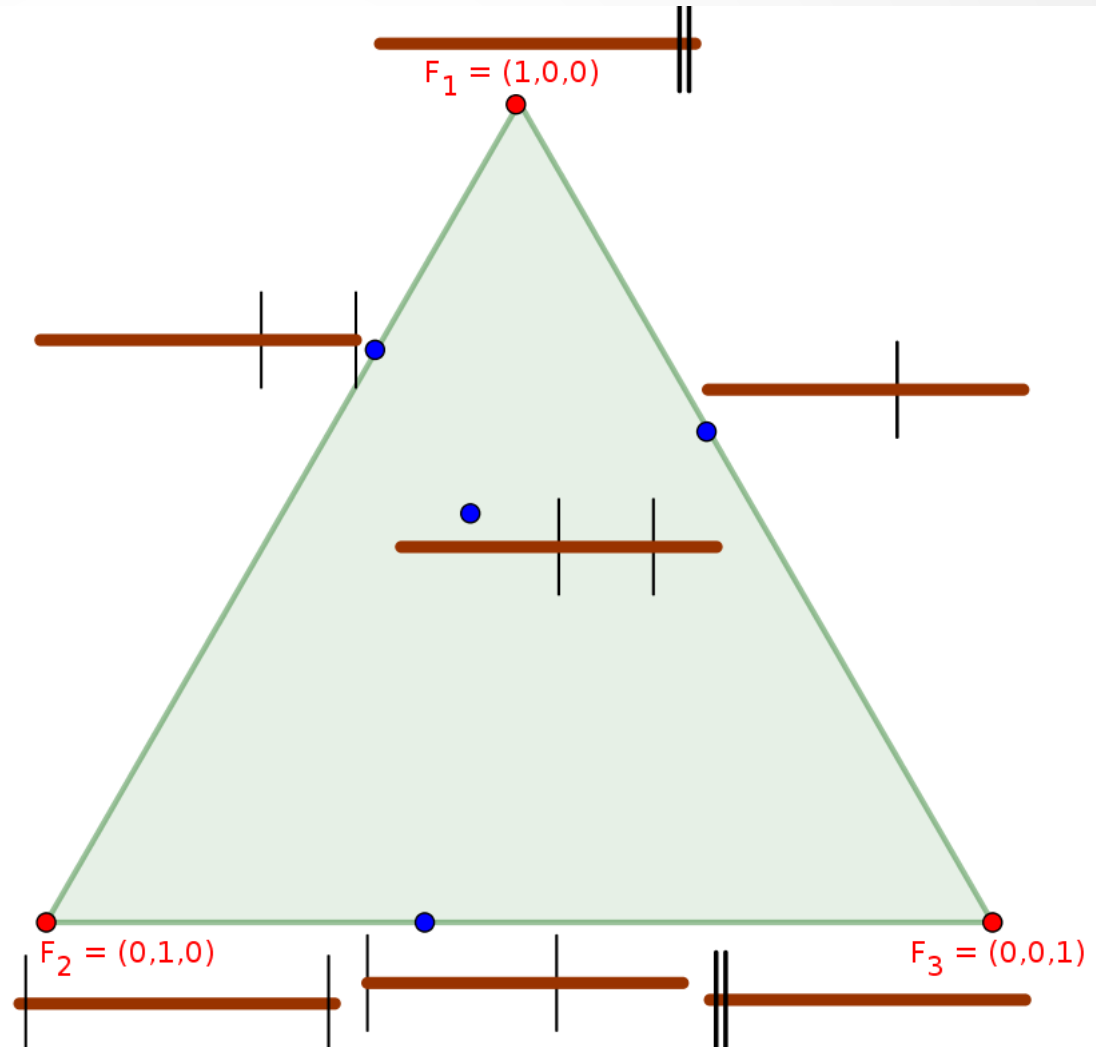
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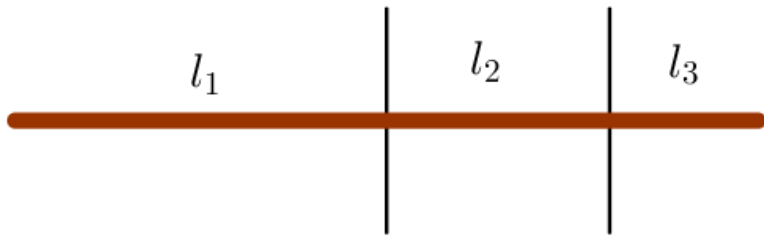
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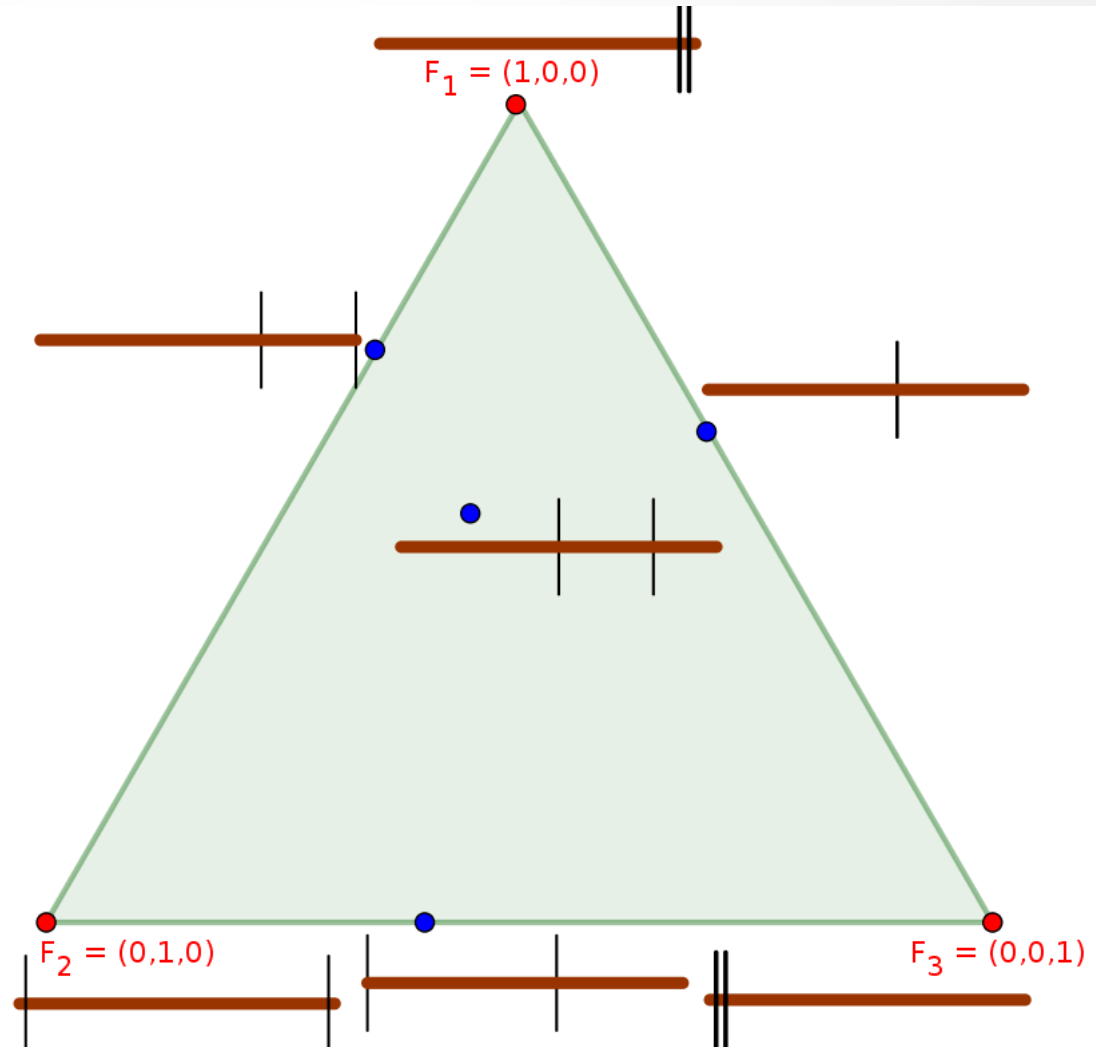


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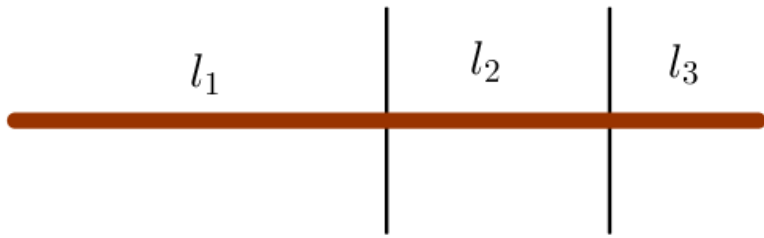
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Envy-free division =



Simplex of Partitions – Definition

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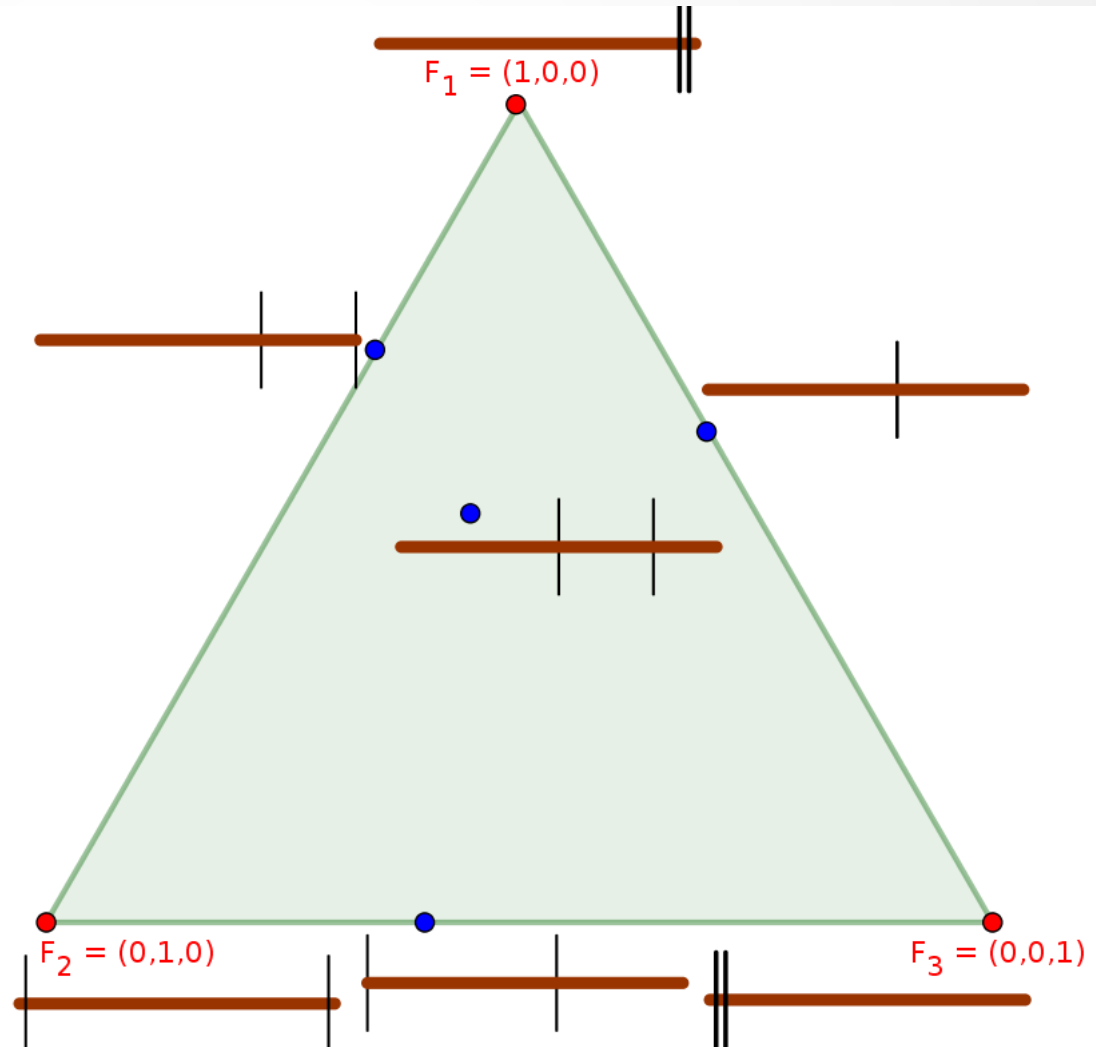


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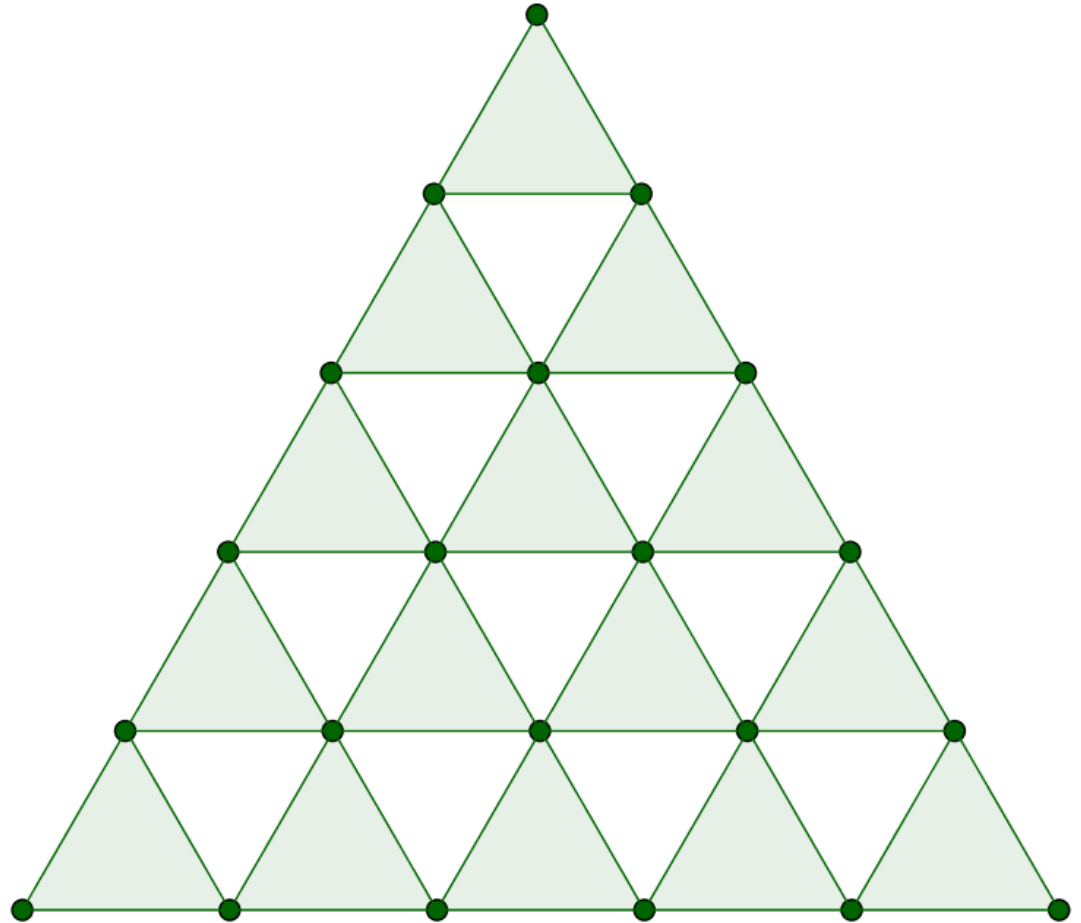
*Envy-free division =
point in which
each agent prefers
a different piece.*



Simplex of Partitions – Triangulation

(based on Simmons 1980, Su 1999)

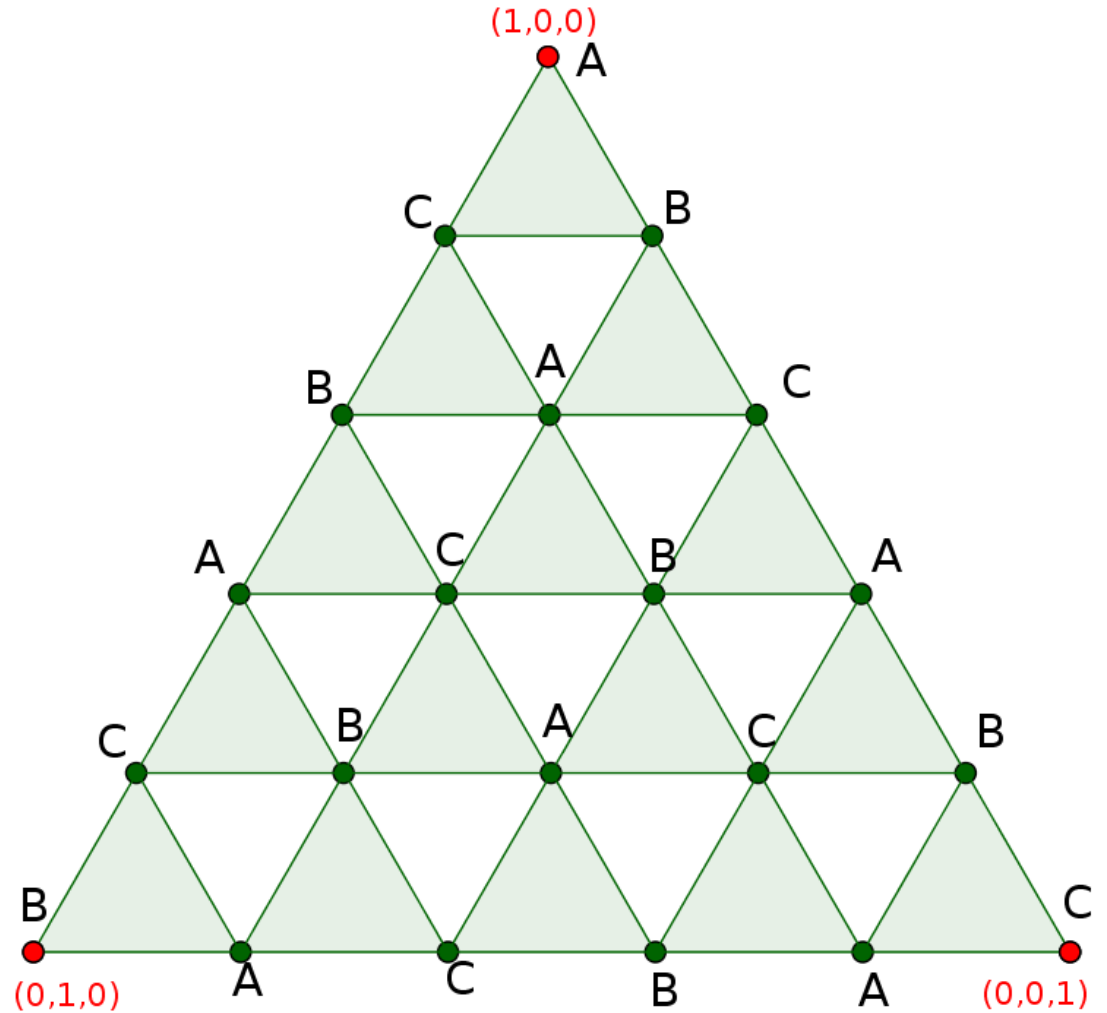
a. Triangulate the simplex.



Simplex of Partitions – Triangulation

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- Triangulate the simplex.
- Assign each vertex to a different agent such that in each sub-simplex, all agents are represented.



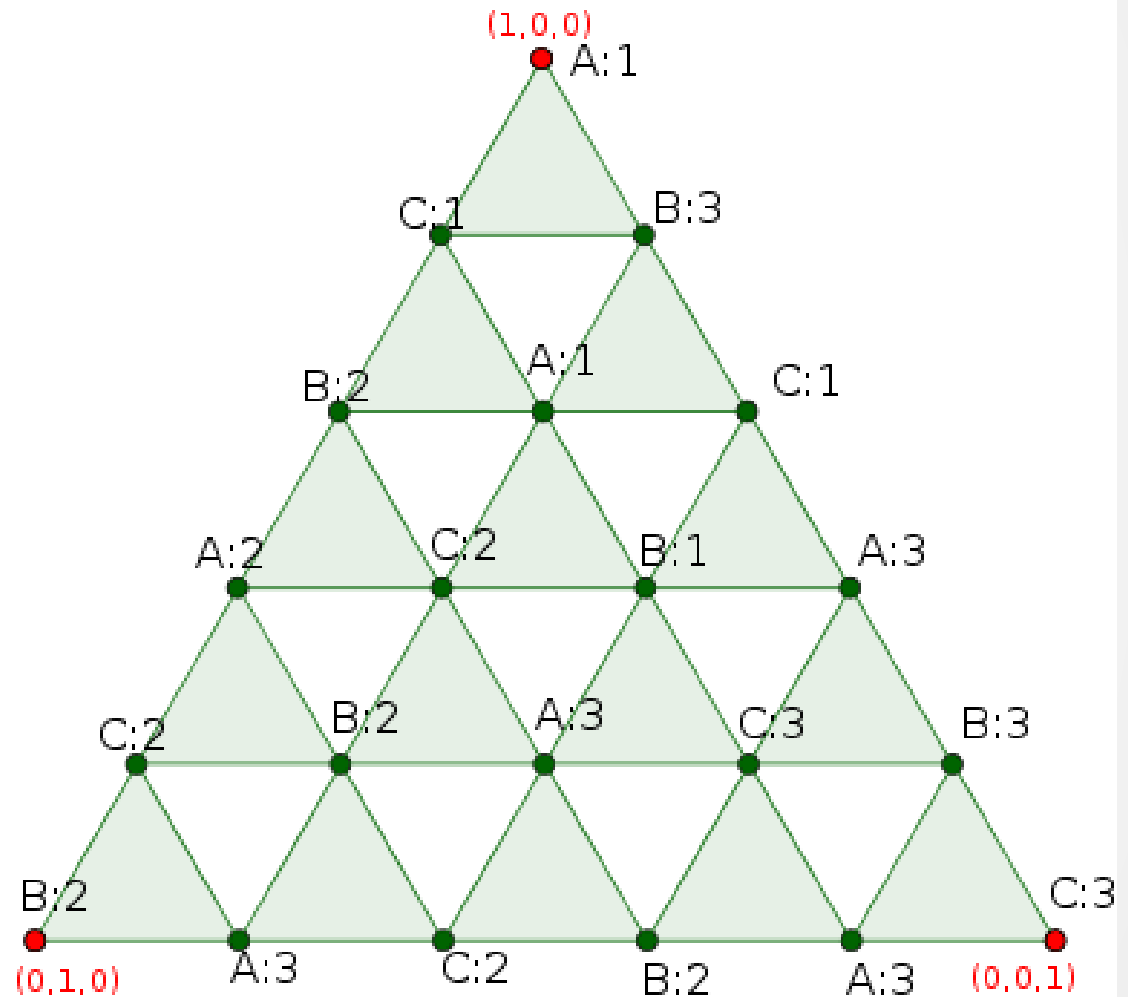
Simplex of Partitions – Triangulation

(based on Simmons 1980, Su 1999)

a. Triangulate the simplex.

b. Assign each vertex to a different agent such that in each sub-simplex, all agents are represented.

c. Ask each agent to label all its vertices by the index of his favorite piece.



Simplex of Partitions – Triangulation

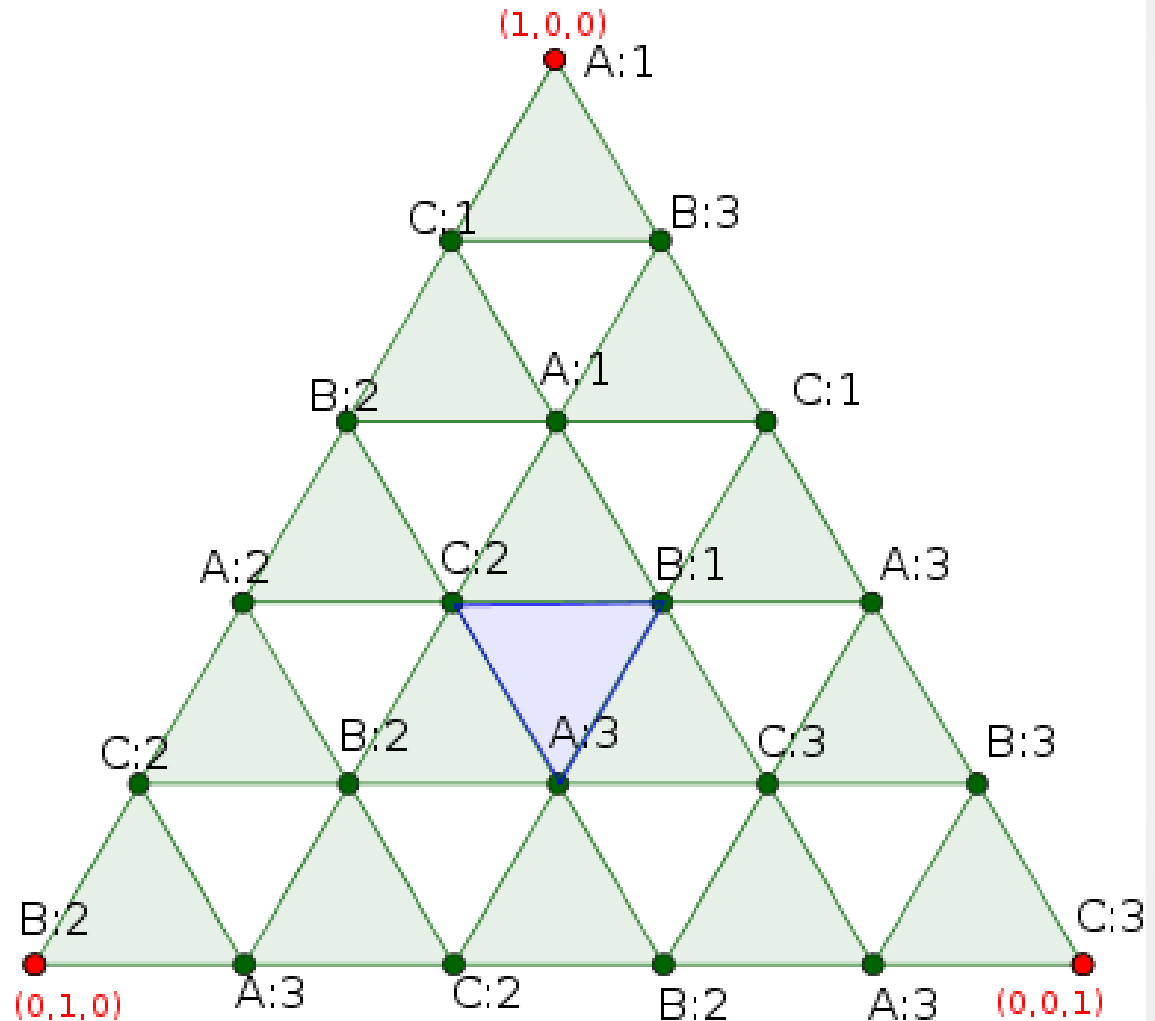
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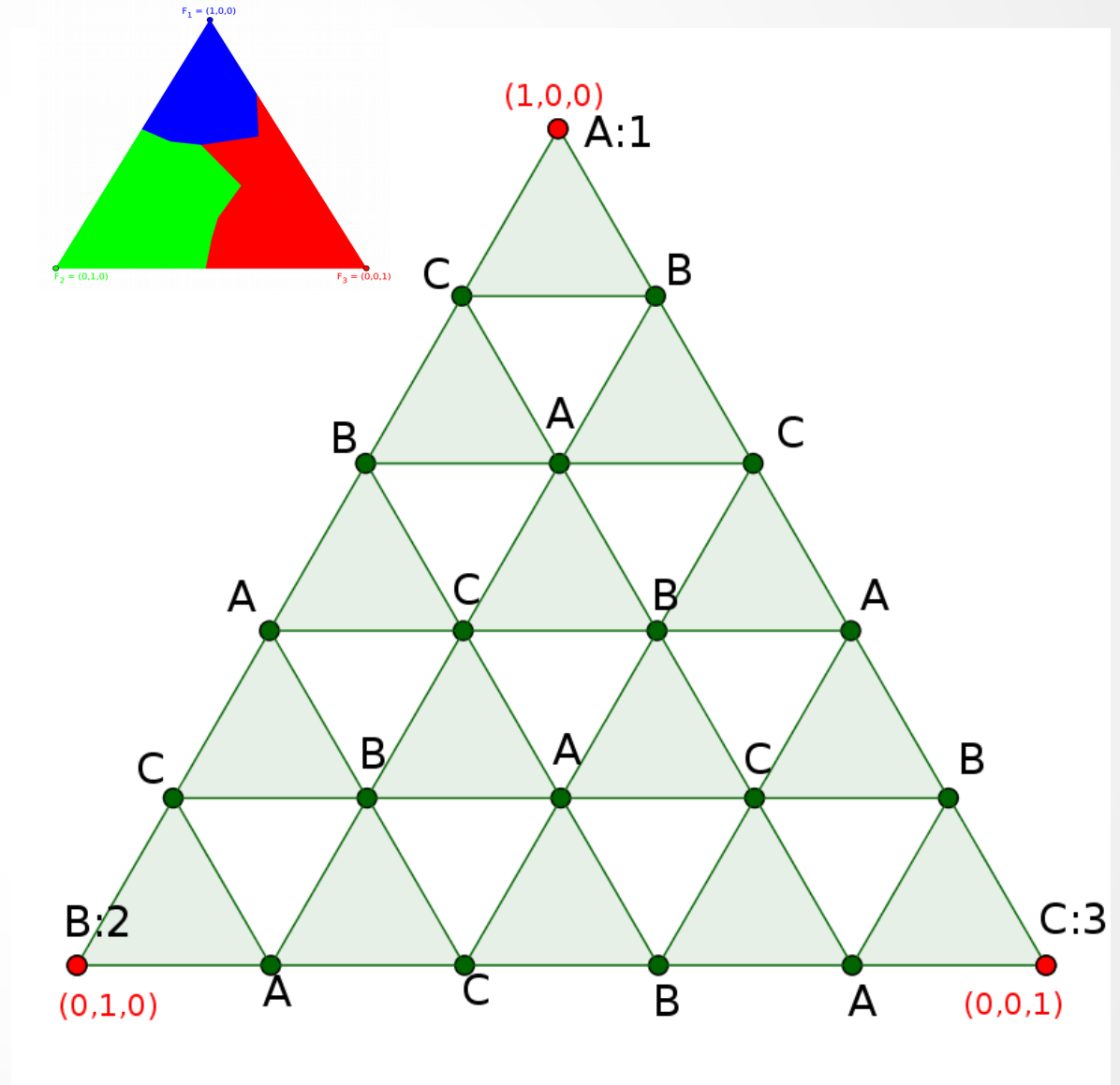
c. Ask each agent to label all its vertices by the index of his favorite piece.

d. A simplex labeled by all n labels = an approximately-envy-free division.



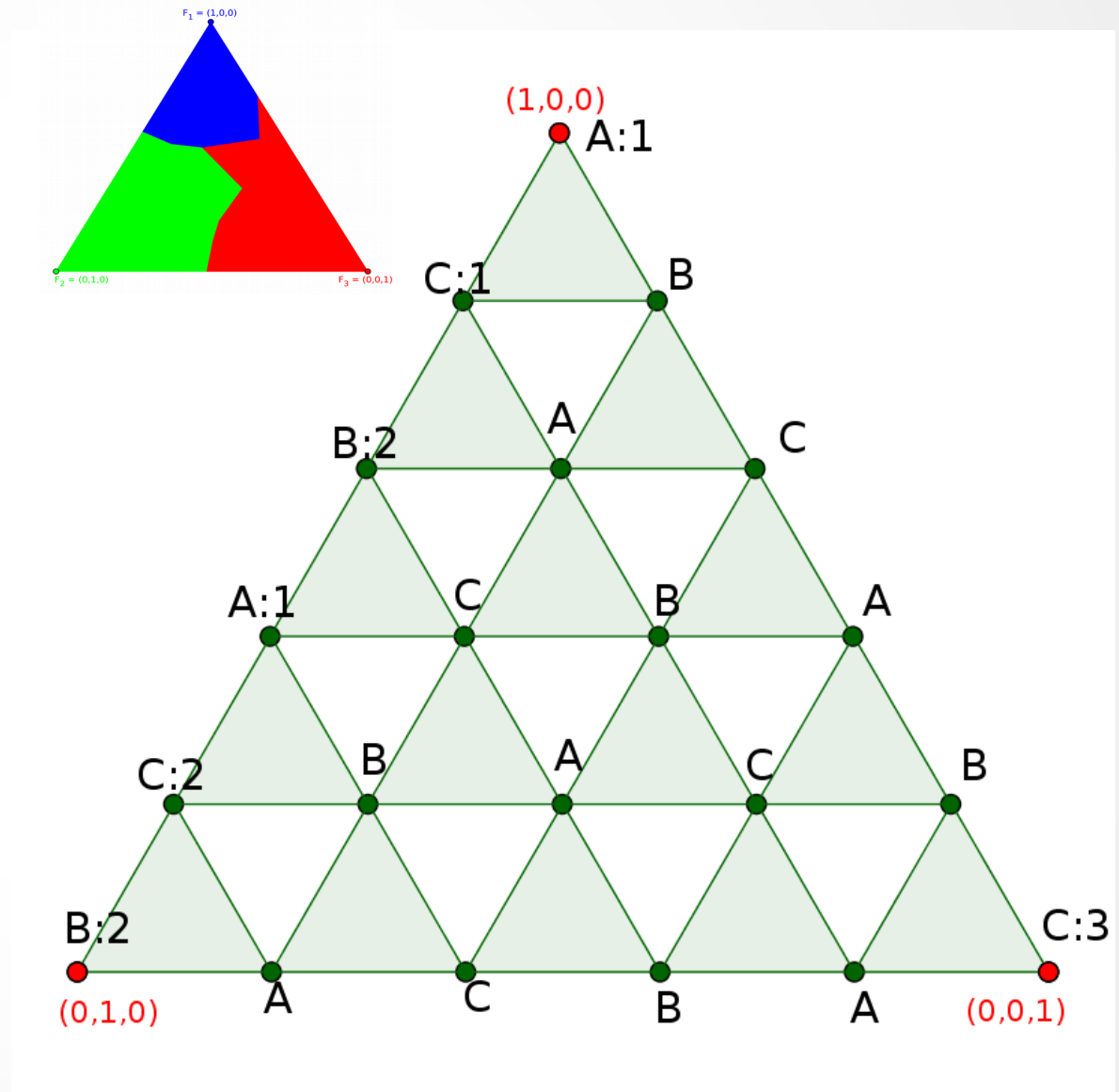
Triangulation – Positive Agents

Fact: When all agents have positive valuations, each face is labeled only with the labels of its endpoints.



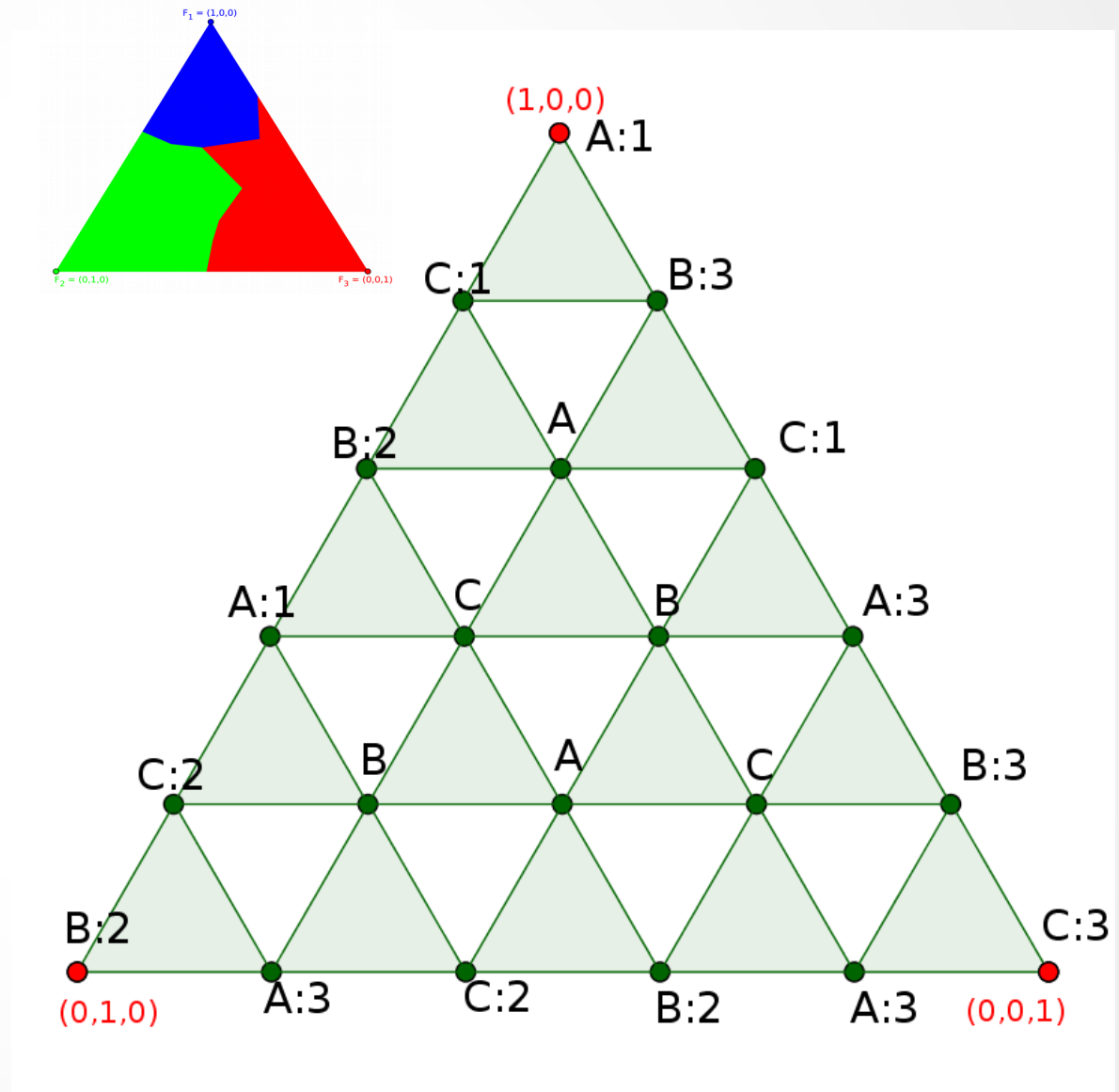
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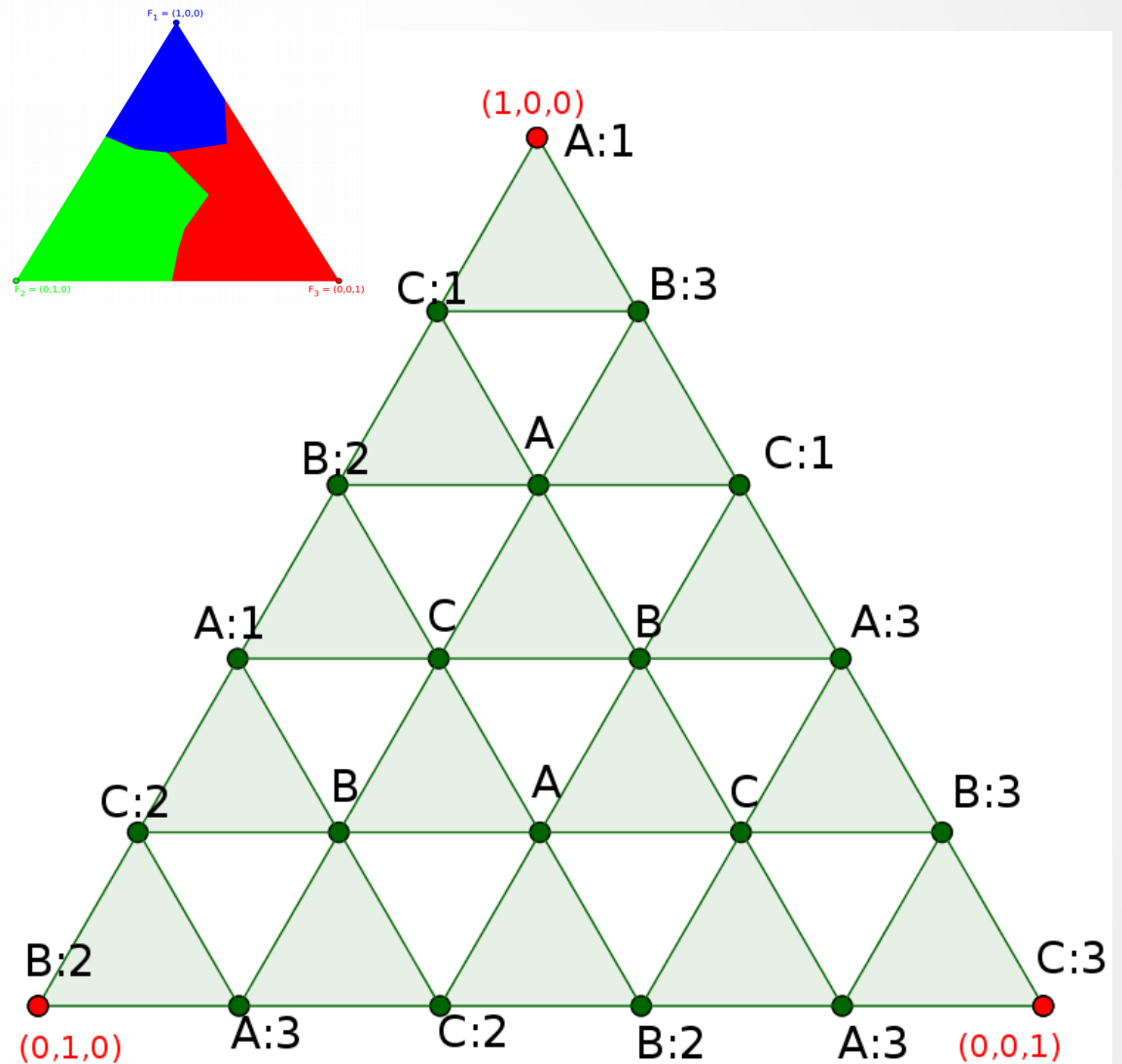
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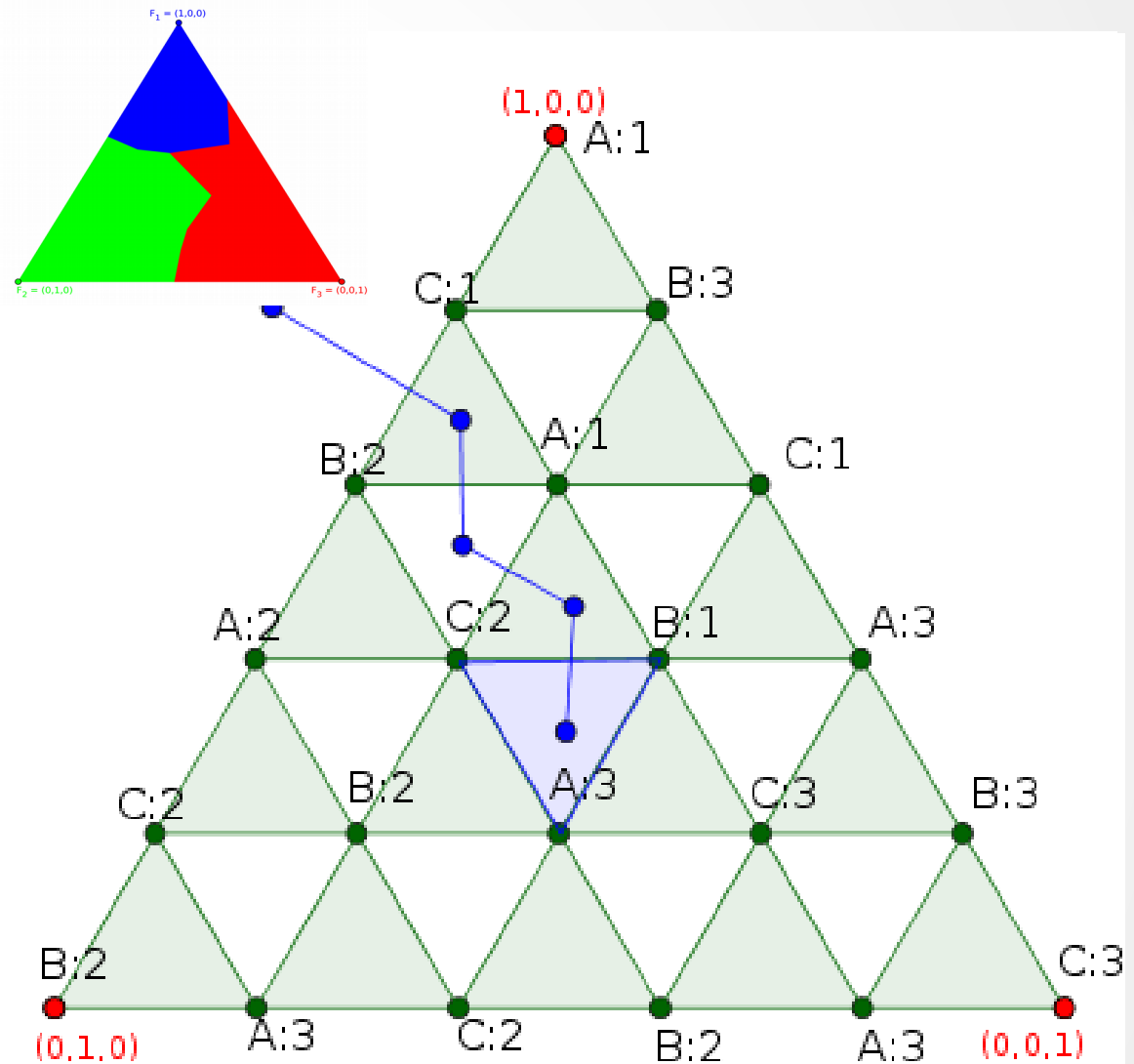
Lemma (Sperner 1929): When each face is labeled only with the labels of its endpoints, a fully-labeled sub-simplex exists.



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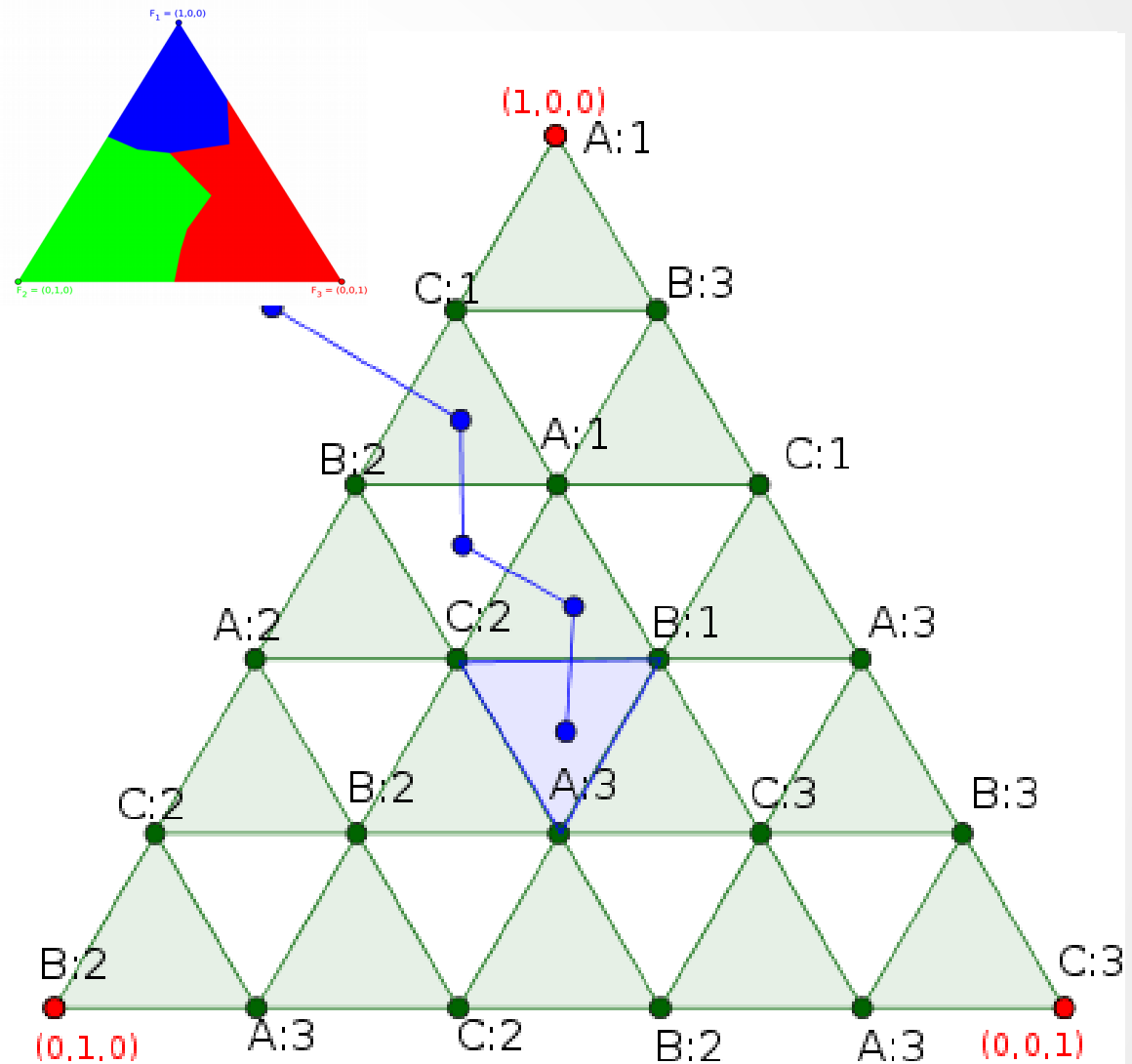
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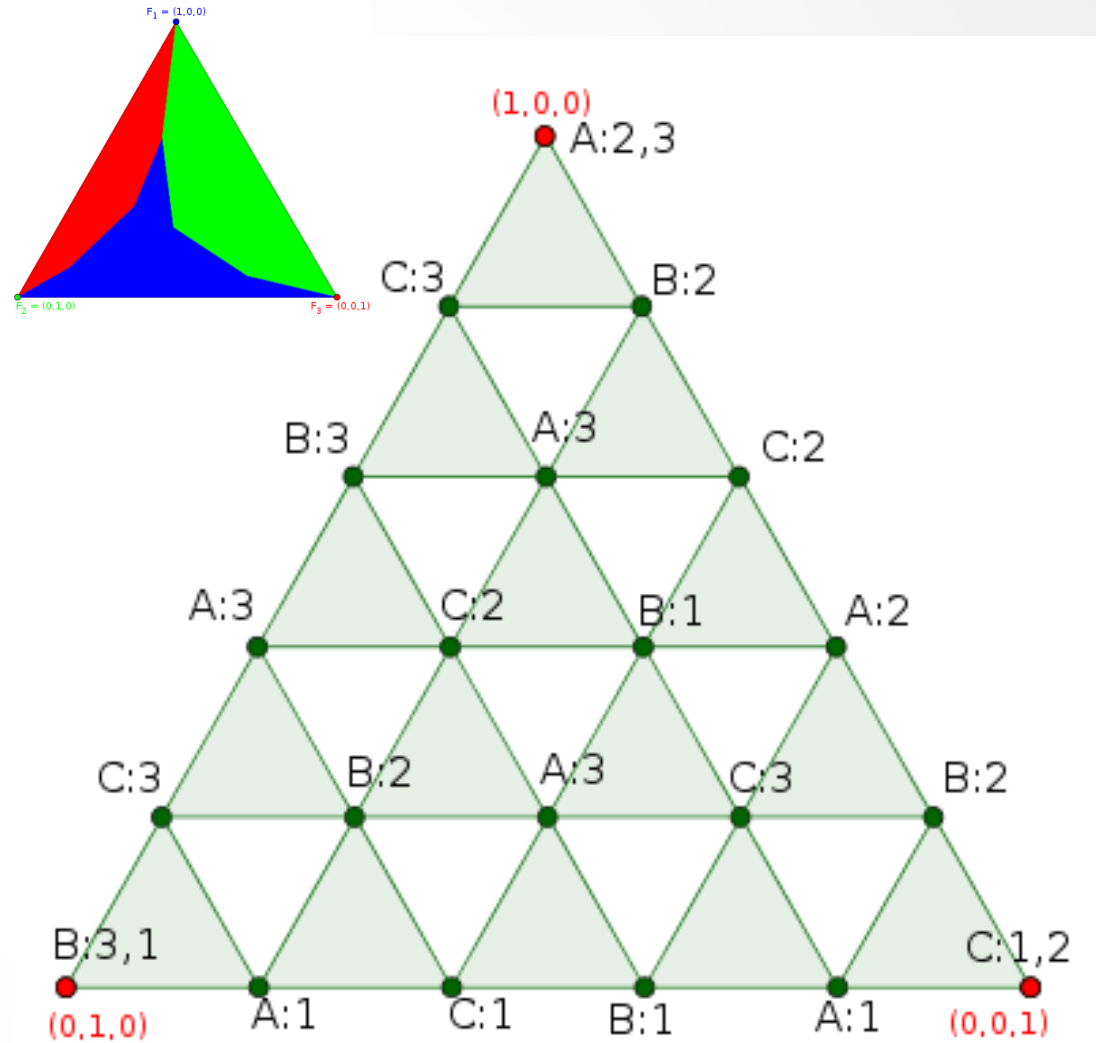
Corollary: when all valuations are positive, an approximately-envy-free division exists.

Corollary (Stromquist 1980, Simmons 1980, Su 1999): when valuations are also continuous, an envy-free division exists.



Triangulation - Negative Agents

Fact: When all agents have negative valuations, it is possible to label the n main vertices such that each face is labeled only with the labels of its endpoints.

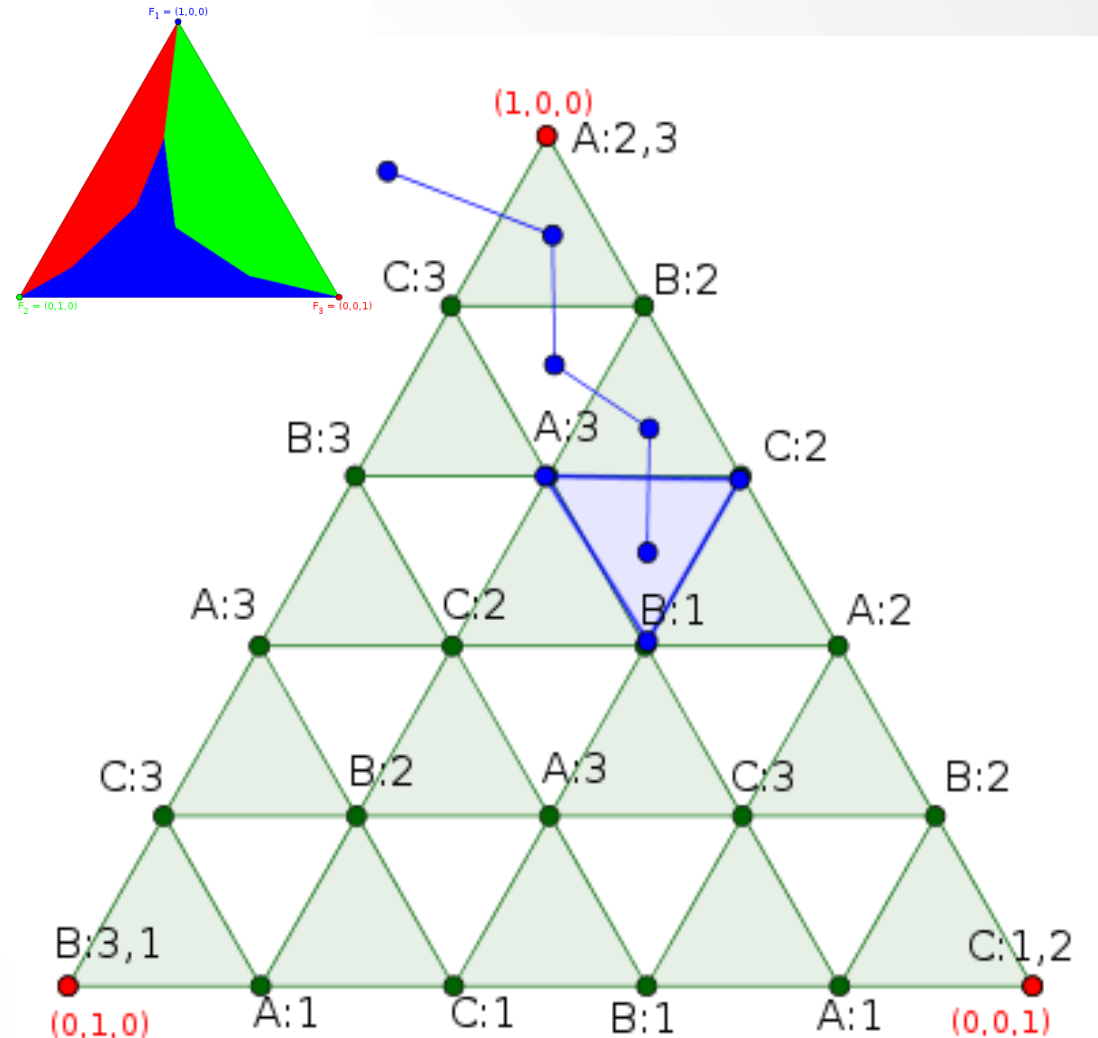


Triangulation - Negative Agents

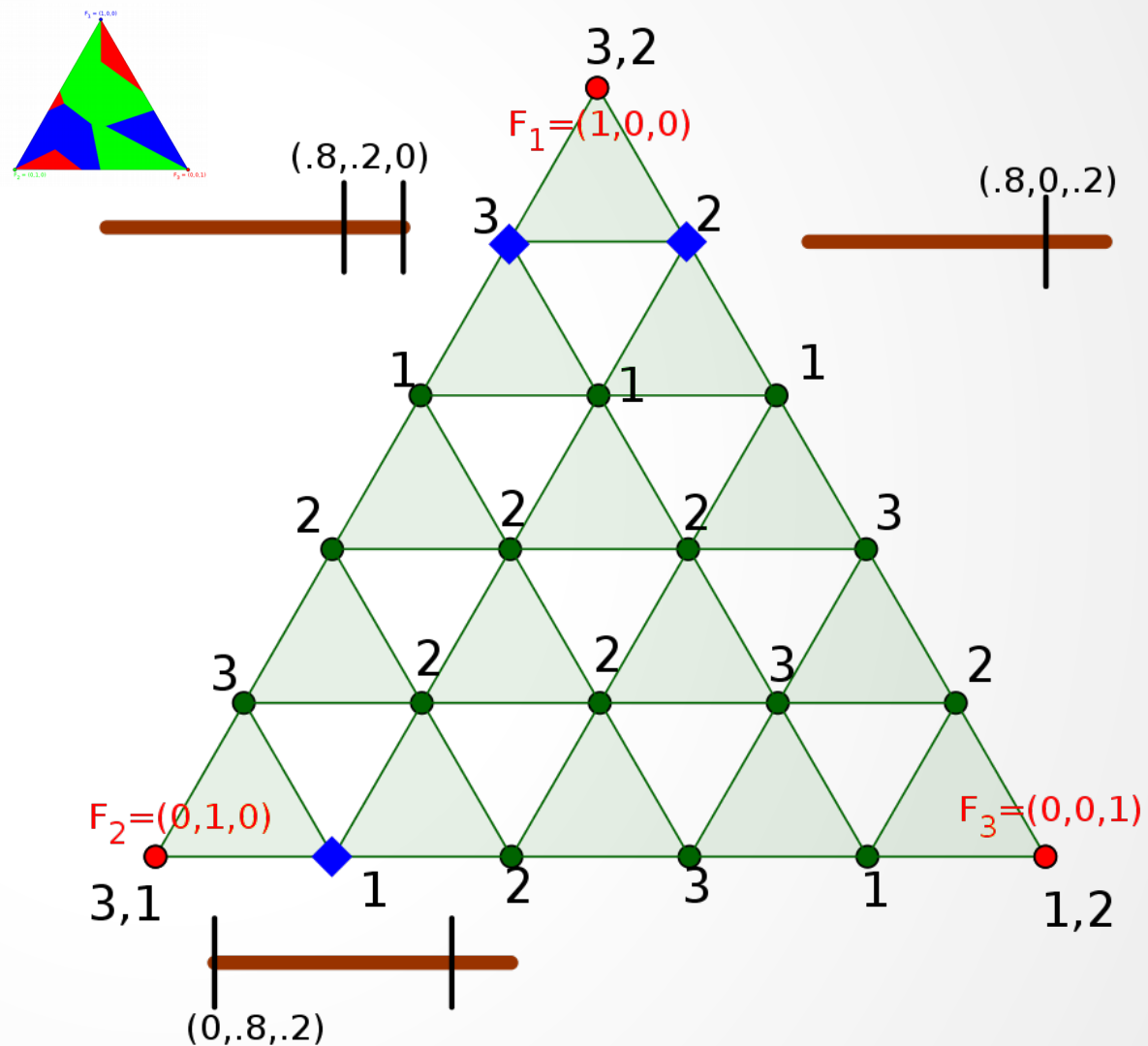
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Corollary (Su 1999): when valuations are also continuous, an envy-free division exists.

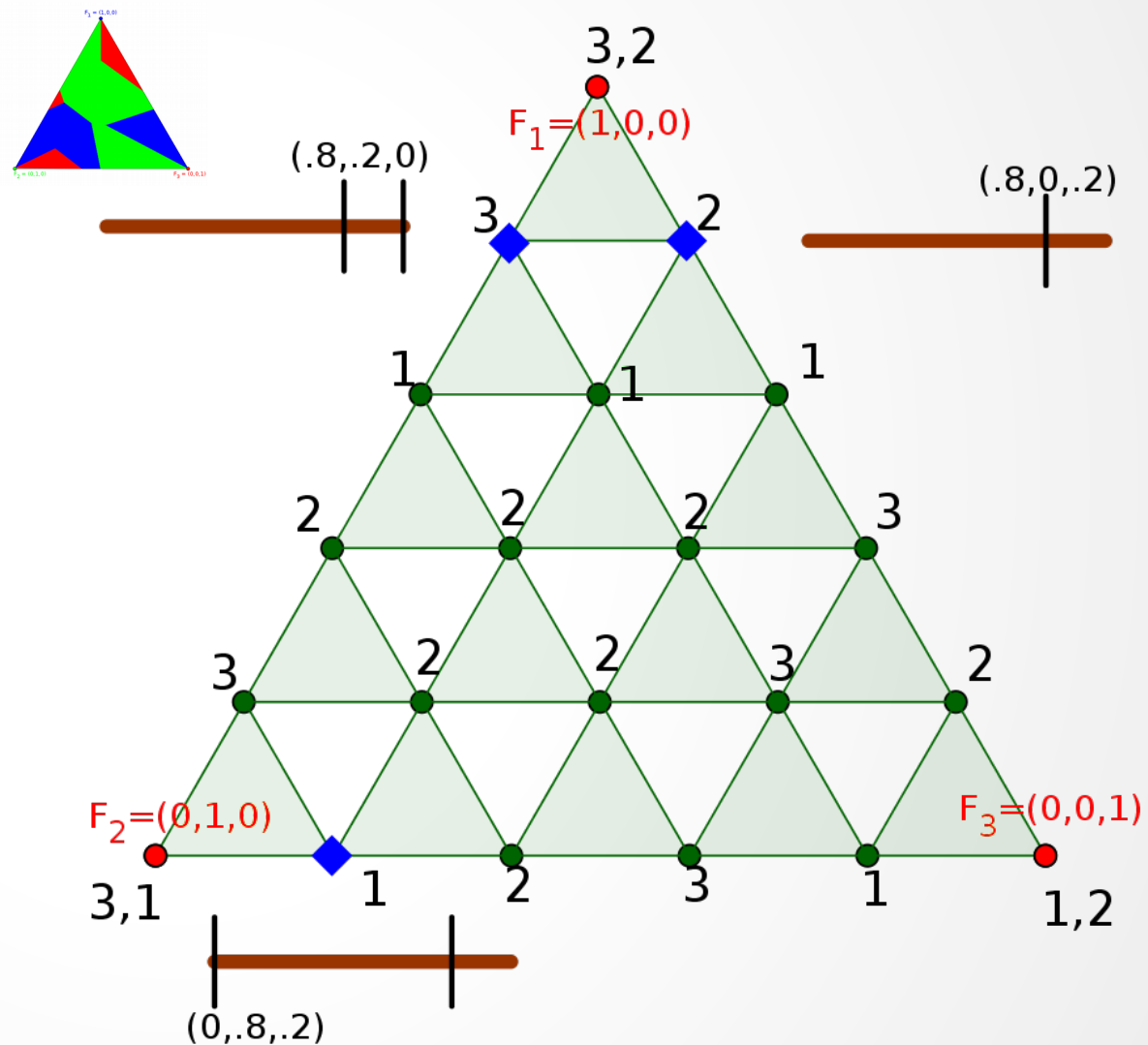


Boundary Permutation Condition



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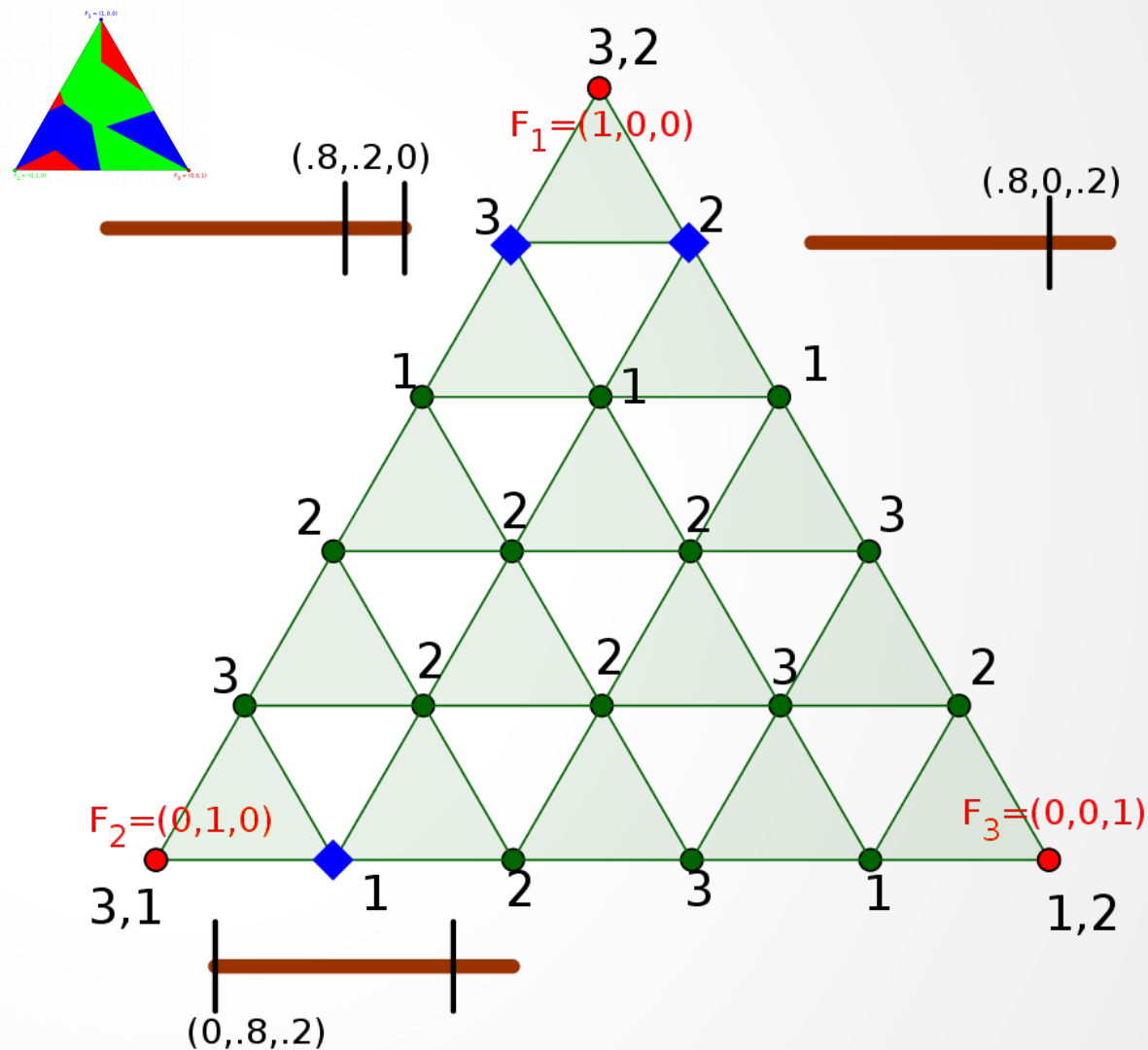
Definition: Two vertices in the simplex are called *friends* if they have the same ordered list of non-zero coordinates.



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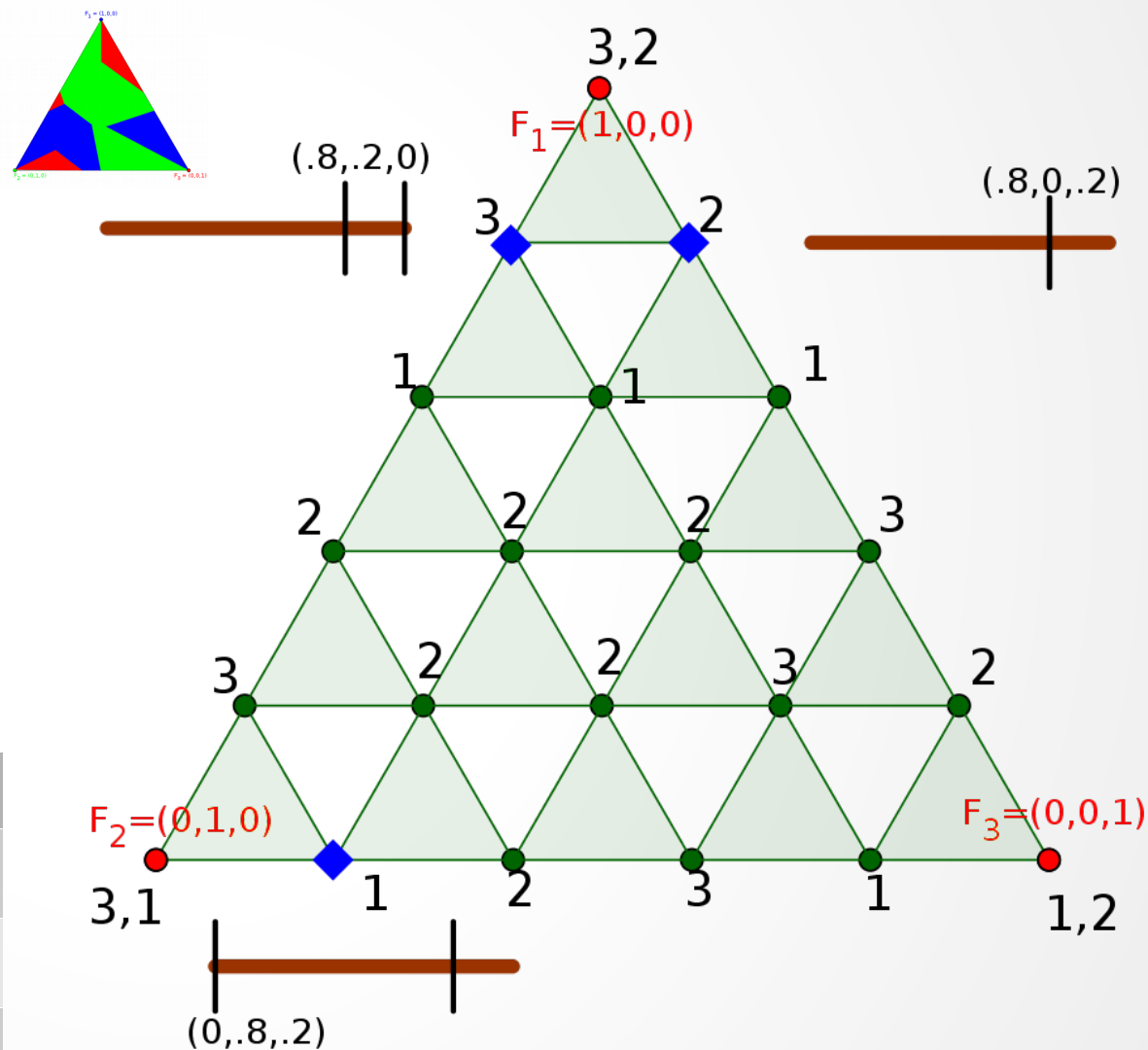
Fact: Each agent's labelings on friends are same up to permutation:



Boundary Permutation Condition

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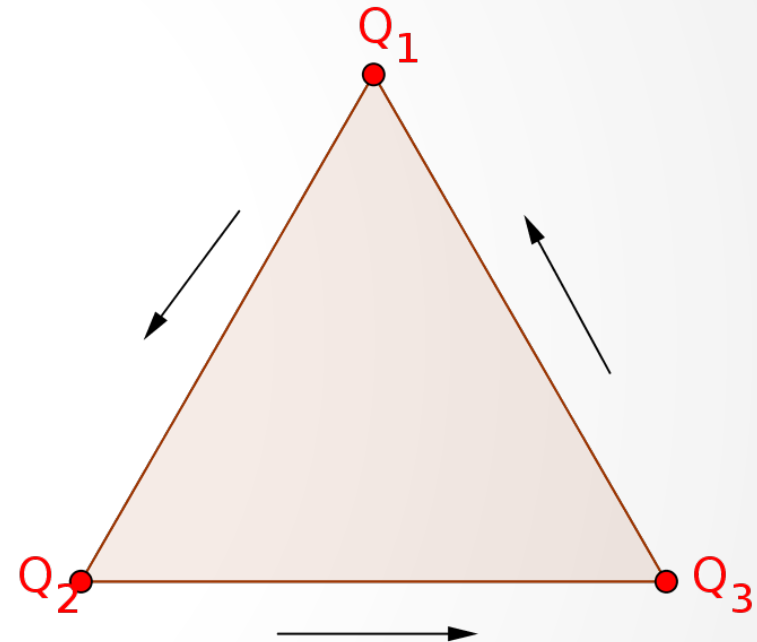
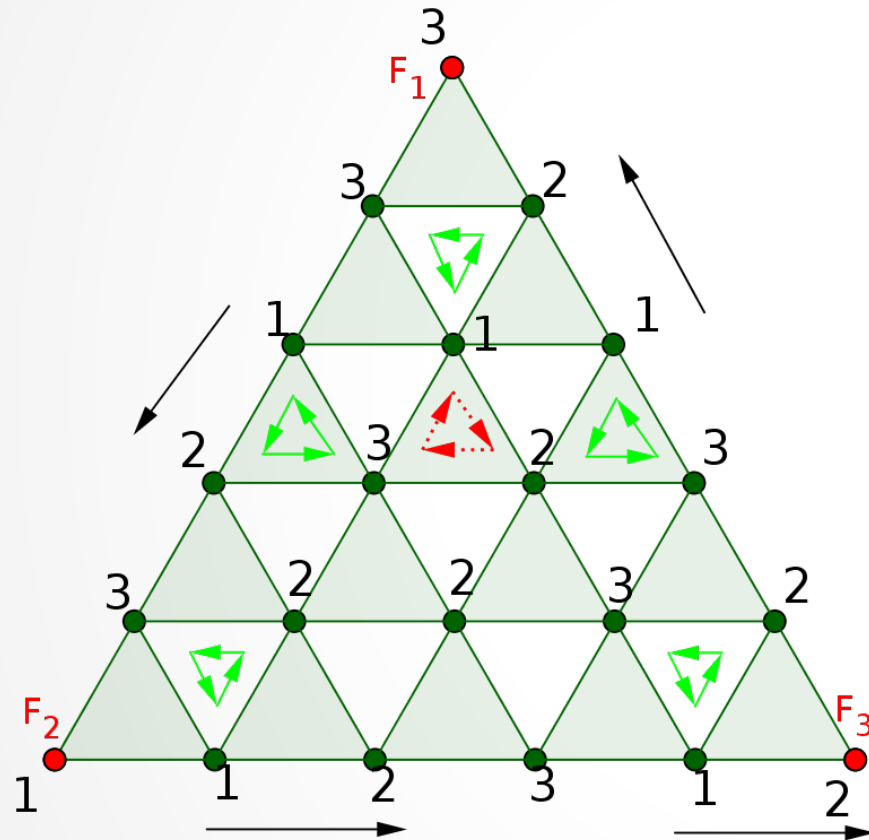
Fact: Each agent's labelings on friends are same up to permutation:



Pref:	Left	Right	Empty	
F_{12}	1	2	3	<i>Even</i>
F_{13}	1	3	2	<i>Odd</i>
F_{23}	2	3	1	<i>Even</i>

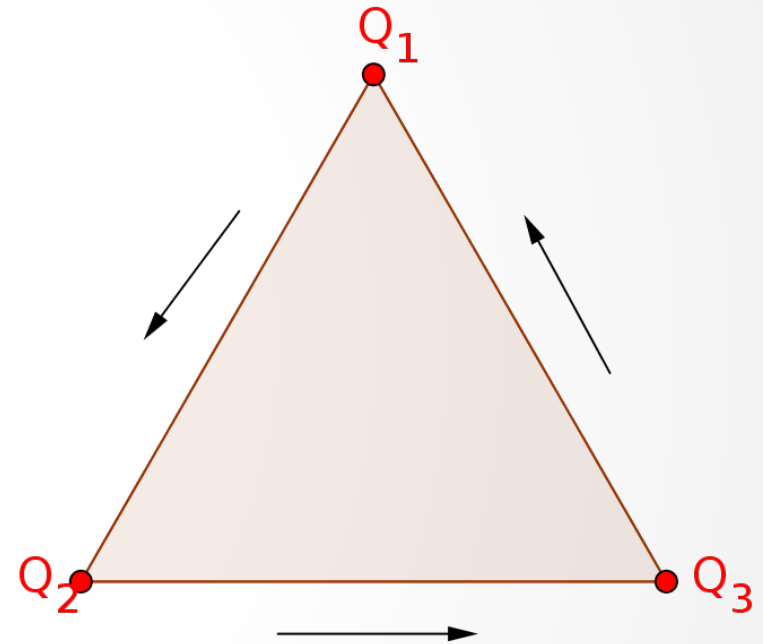
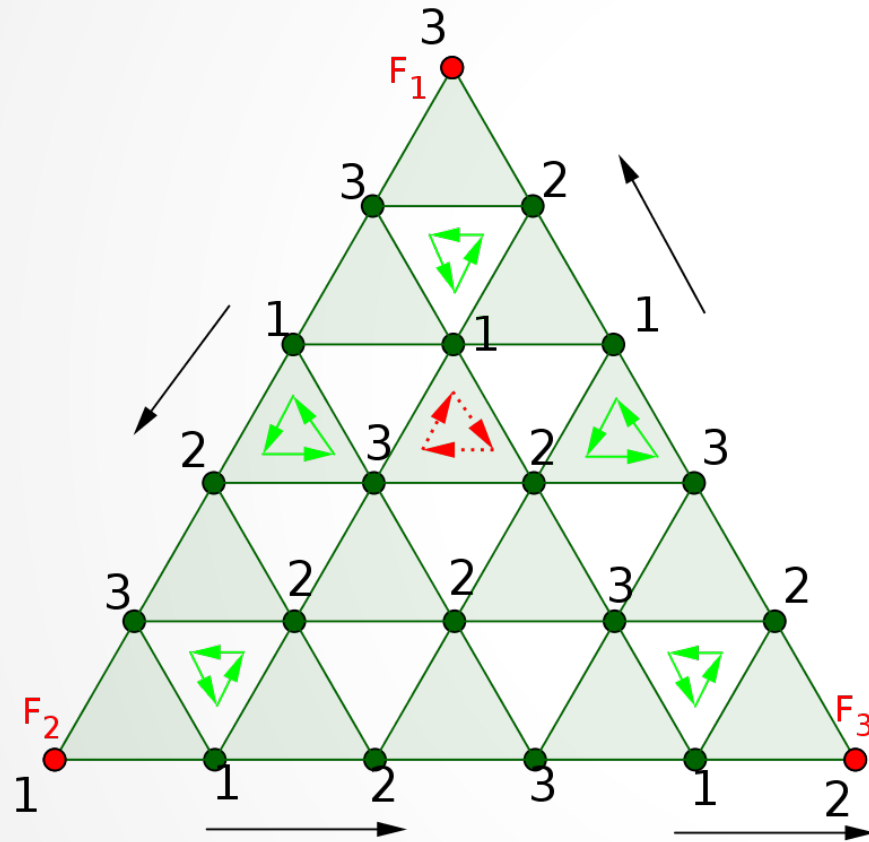
Degree of Labeling

Labeling \equiv mapping from triangulation vertices to vertices of Q
(follows Musin 2014)



Degree of Labeling

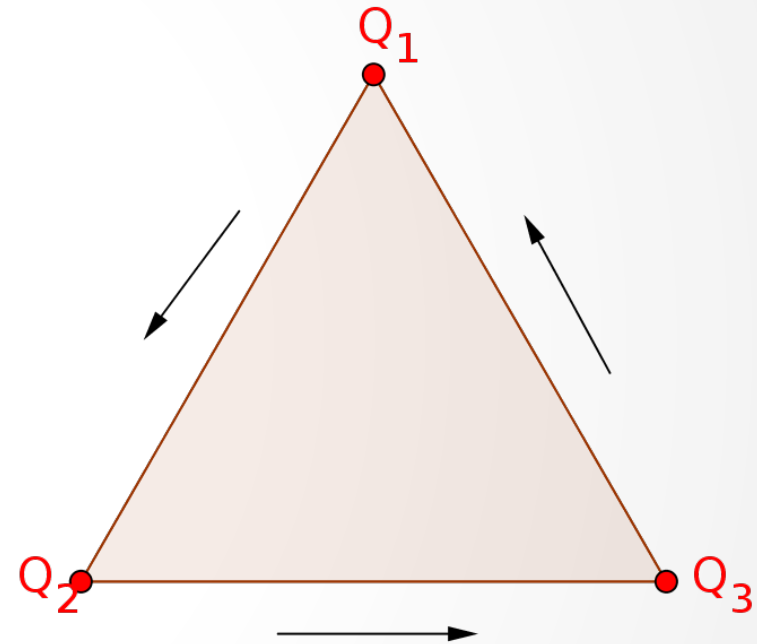
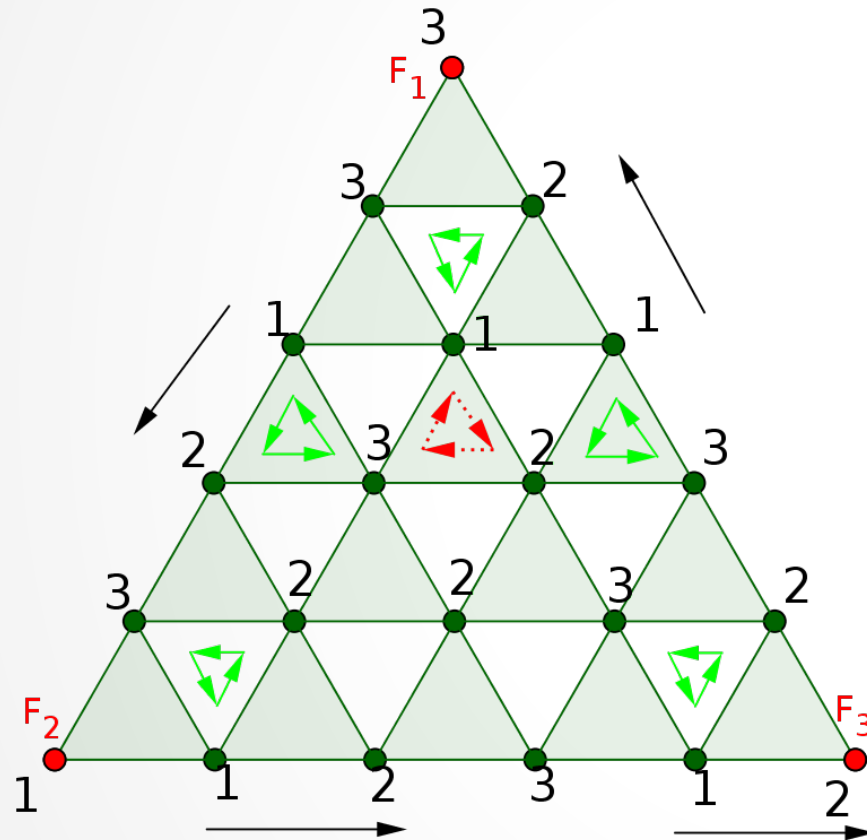
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Degree of mapping = net number of rounds (*CCW=positive*).
Lemma: degree on boundary = degree in interior.

Steps in Existence Proof

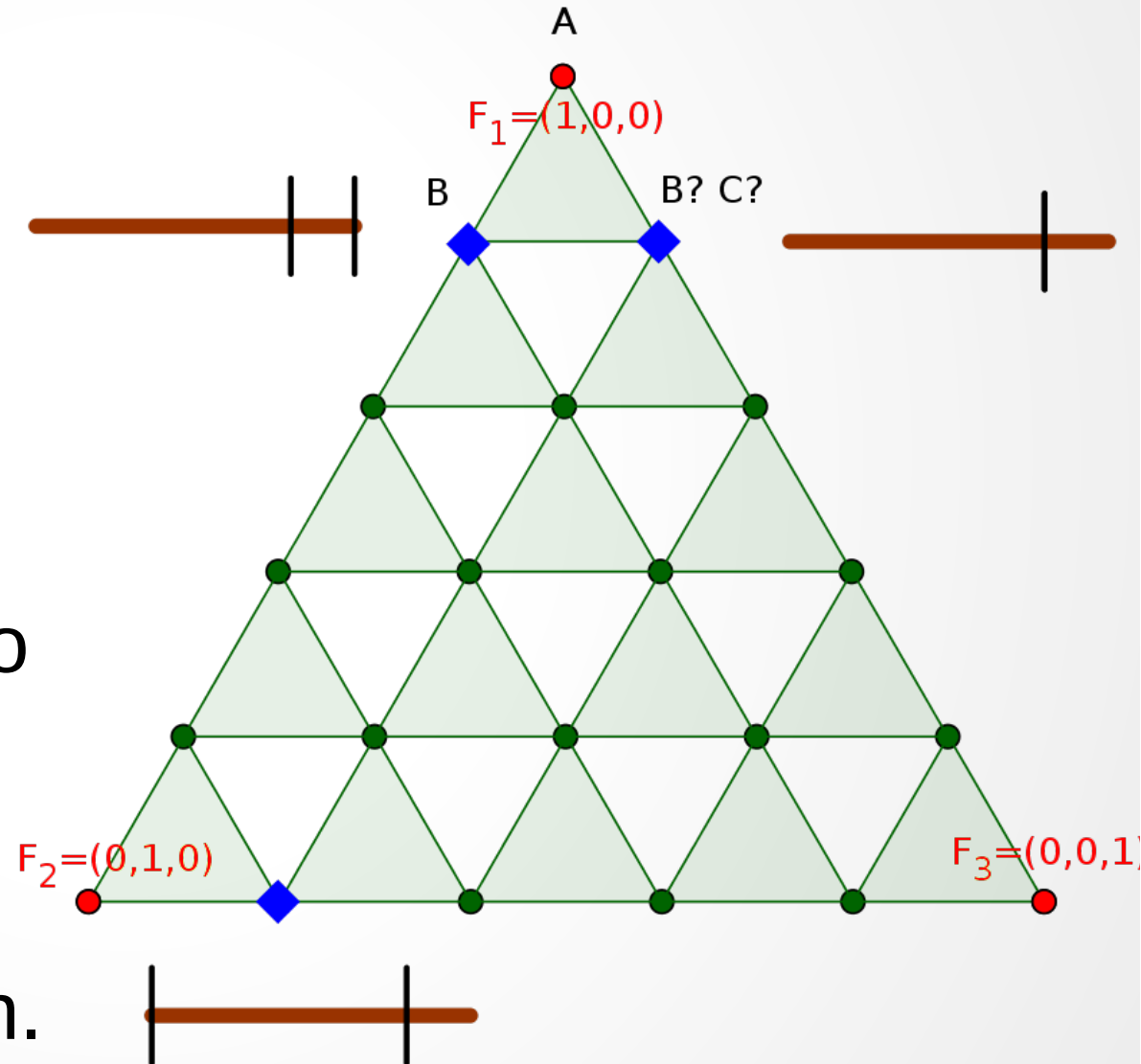
Step	Proved for
1. n agent-labelings with permutation condition → Combined labeling with permutation condition	Any n
2. Permutation condition → Nonzero boundary degree	$n = 3$
3. Boundary degree = Interior degree	Any n (?)

Step 1: n labelings \rightarrow 1 labeling

We need to assign owners to vertices s.t.:

- In each sub-simplex, each vertex belongs to a **different** owner.
- Friends are assigned to **the same** owner.

Does not work with the equilateral triangulation.



Step 1: n labelings \rightarrow 1 labeling

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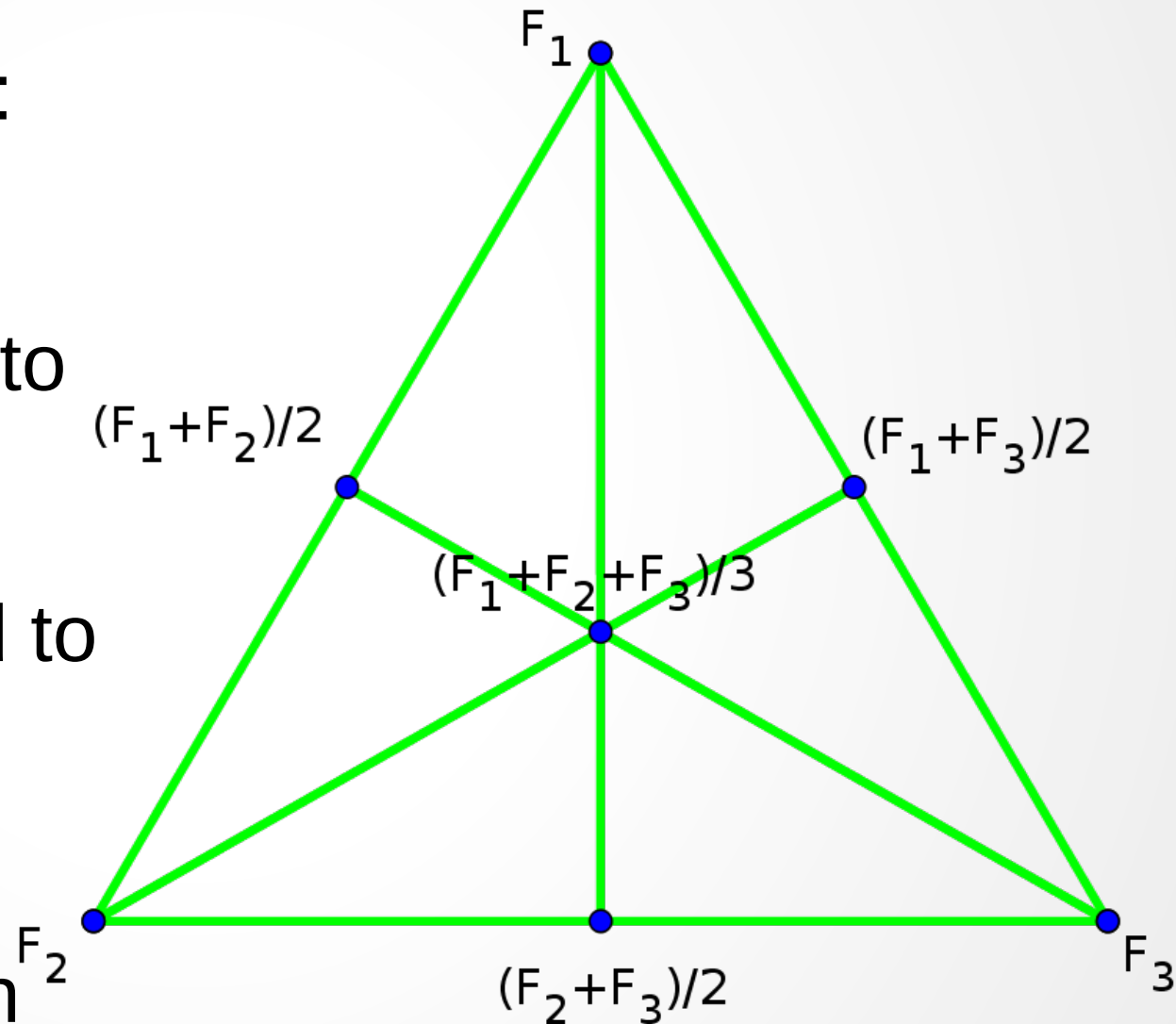
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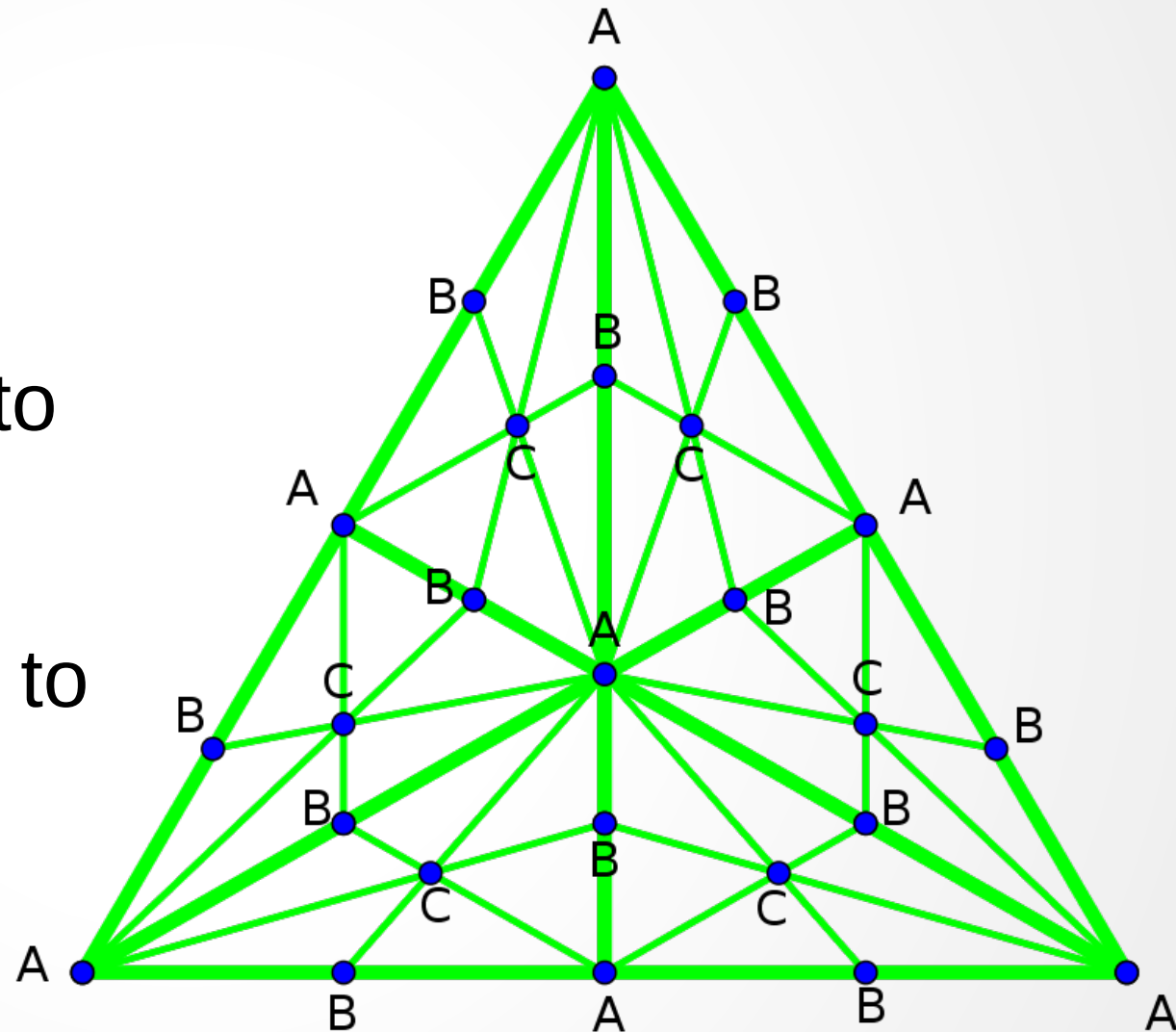


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Step 2: Permutation → Boundary degree

Permutation condition:

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Agent condition: Either:
(+) In each main-vertex i , the label is i , or:
(-) In each main-vertex i , the label can be anything but i .

Lemma: *When $n=3$, if labeling satisfies permutation condition and agent condition, then labels on main vertices can be chosen such that: boundary-degree mod 3 \neq 0.*

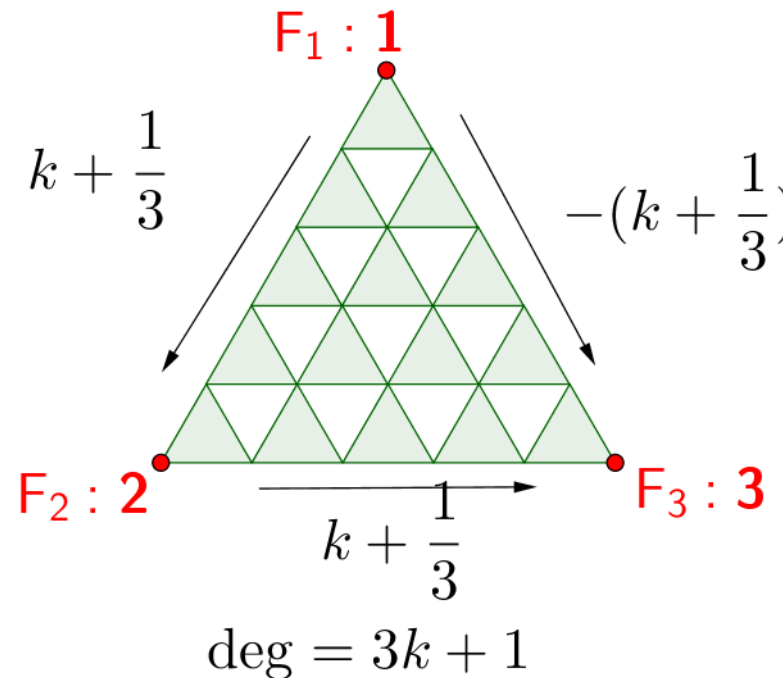
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Proof:



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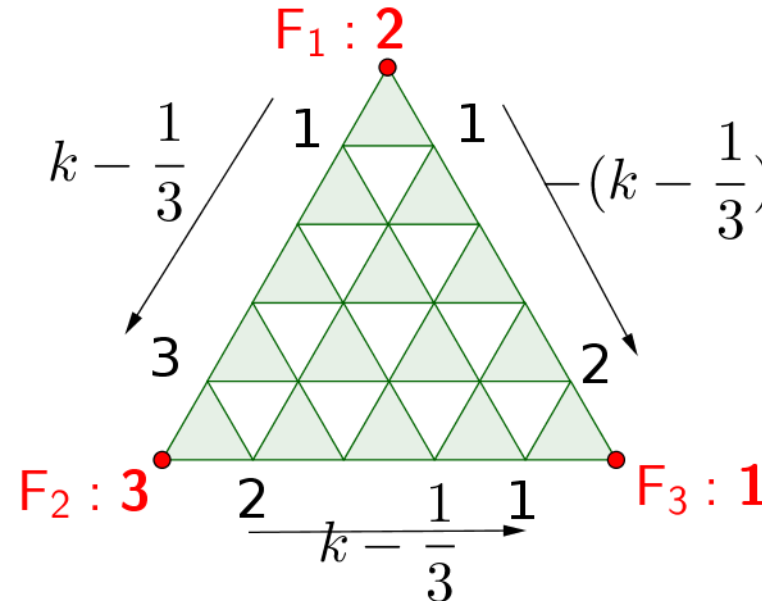
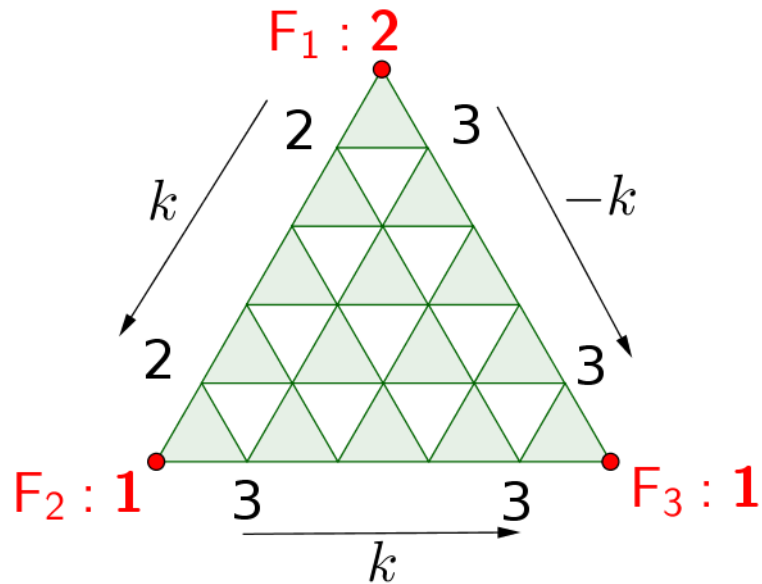
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2 of 9 cases shown below:



$$\text{deg} = (3k) - 2/3 - 1/3 = 3k - 1 \quad \text{deg} = (3k - 1) - 1/3 + 1/3 = 3k - 1$$

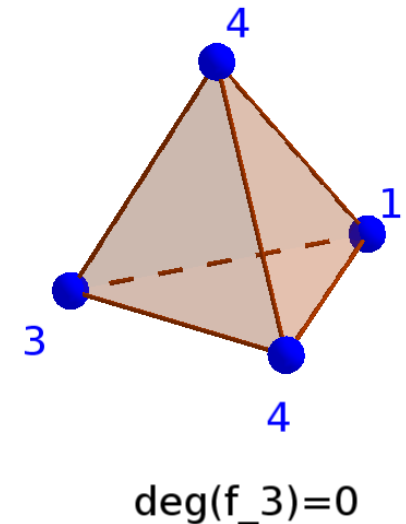
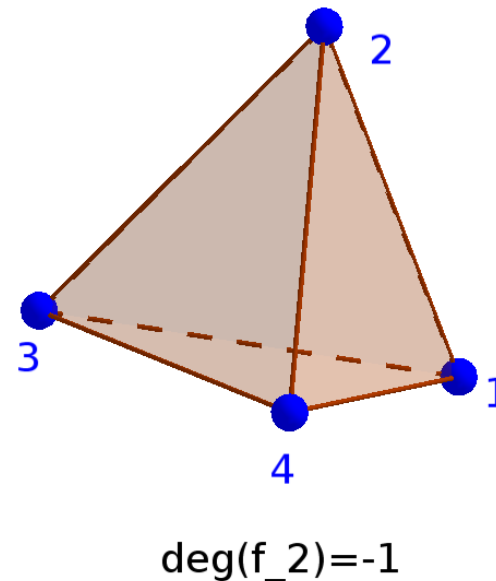
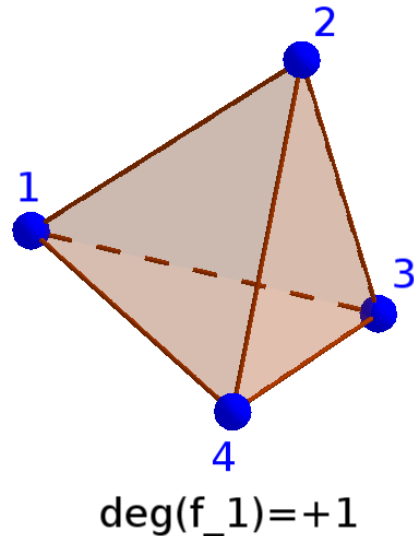
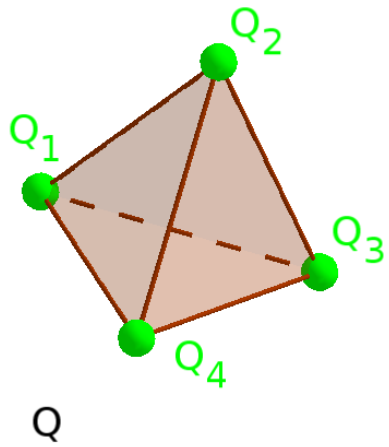
Step 3: Boundary degree = Interior degree

Definition:

Degree of labeling of an n -simplex in R^{n-1}

= sign of determinant of affine transformation to Q

= +1 if onto&no reflection, -1 if onto&one reflection,
0 if not onto.



Step 3: Boundary degree = Interior degree

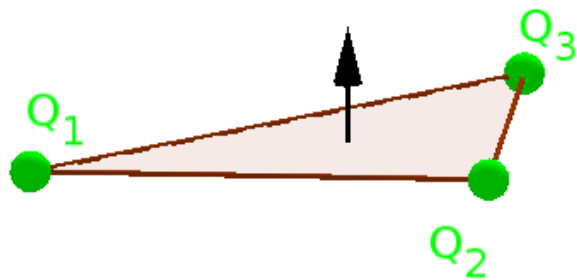
Definition:

Orientation of an $(n-1)$ -simplex in R^{n-1}

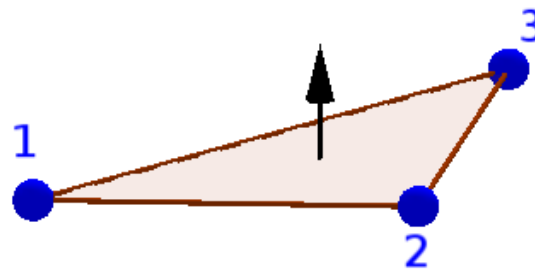
= one of its two adjacent half-spaces.

Degree of labeling of an $(n-1)$ -simplex in R^{n-1}

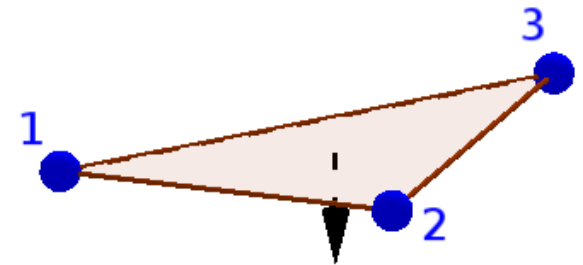
= sign of determinant of *any* affine transformation to Q that preserves the orientation.



Q



$\deg(f_1) = +1$



$\deg(f_2) = -1$

Step 3: Boundary degree = Interior degree

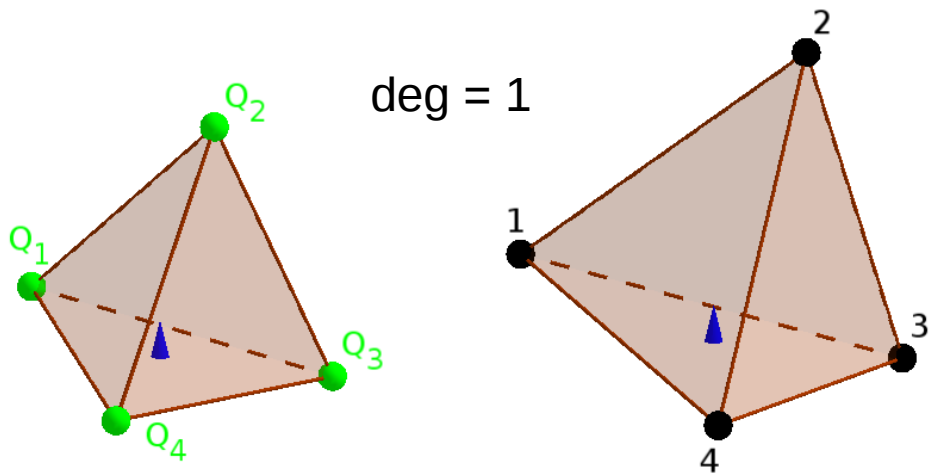
Lemma:

Degree of a labeling of an n -simplex in R^{n-1} ,
= sum of degrees on each face oriented *inwards*:

Step 3: Boundary degree = Interior degree

Lemma:

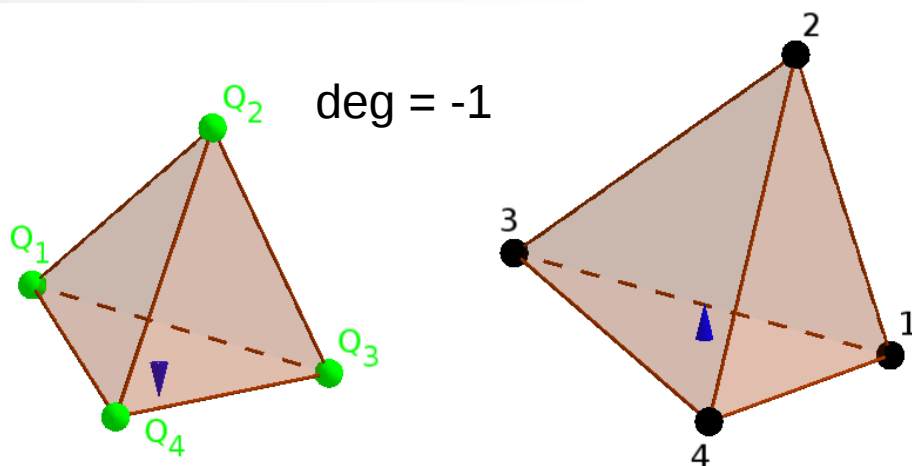
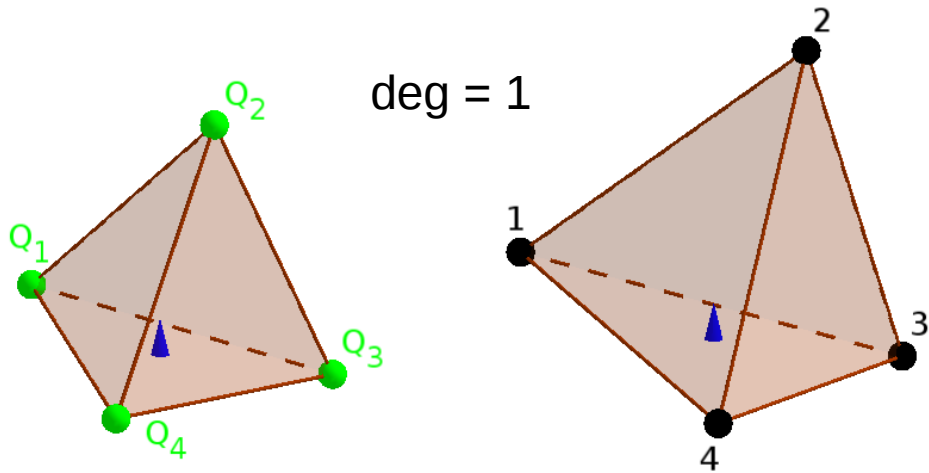
Degree of a labeling of an n -simplex in R^{n-1} ,
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Step 3: Boundary degree = Interior degree

Lemma:

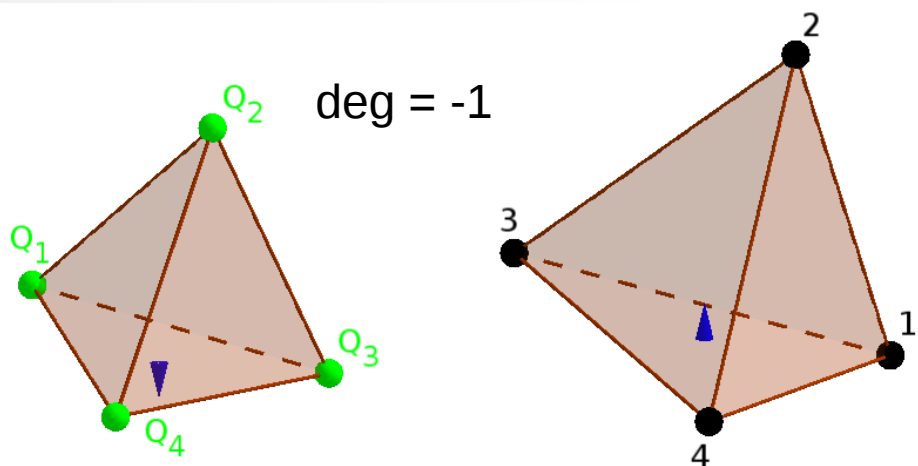
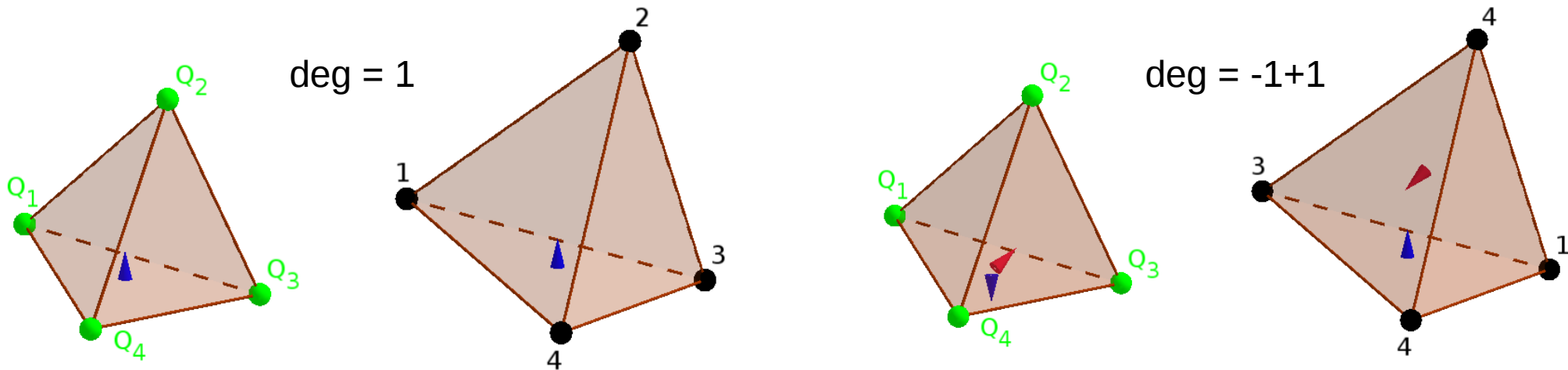
Degree of a labeling of an n -simplex in R^{n-1} ,
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Step 3: Boundary degree = Interior degree

Lemma:

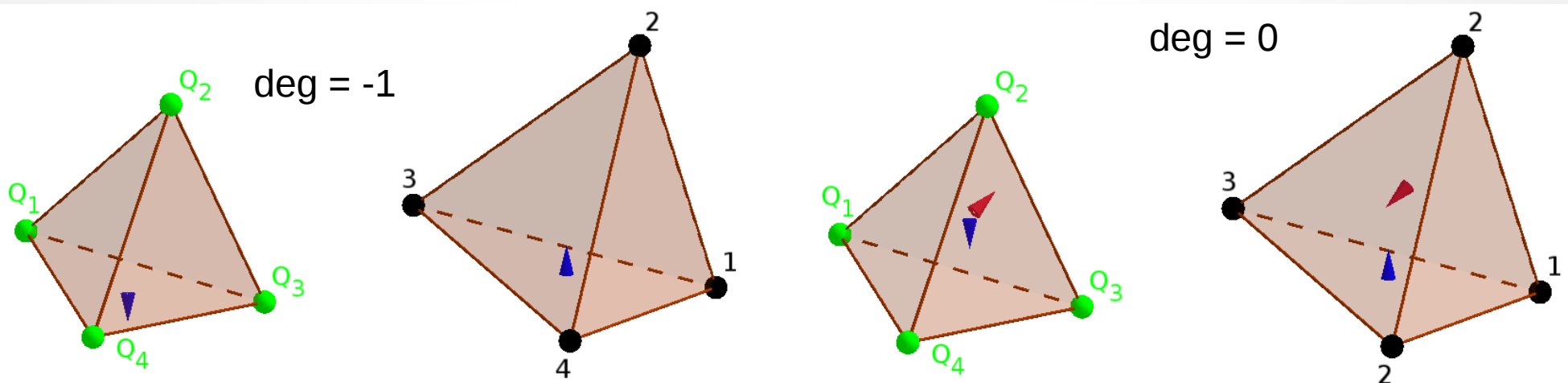
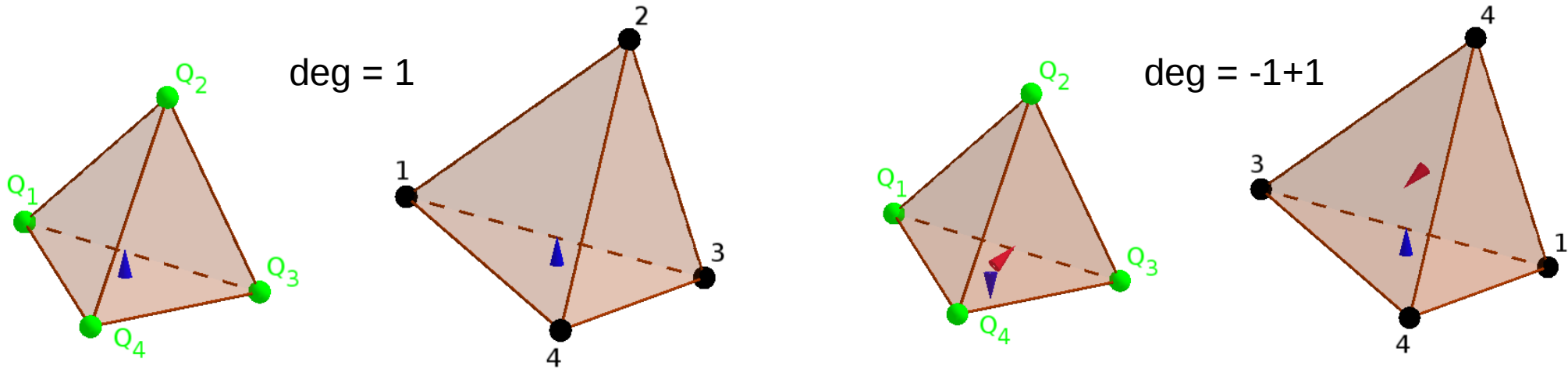
Degree of a labeling of an n -simplex in R^{n-1} ,
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Step 3: Boundary degree = Interior degree

Lemma:

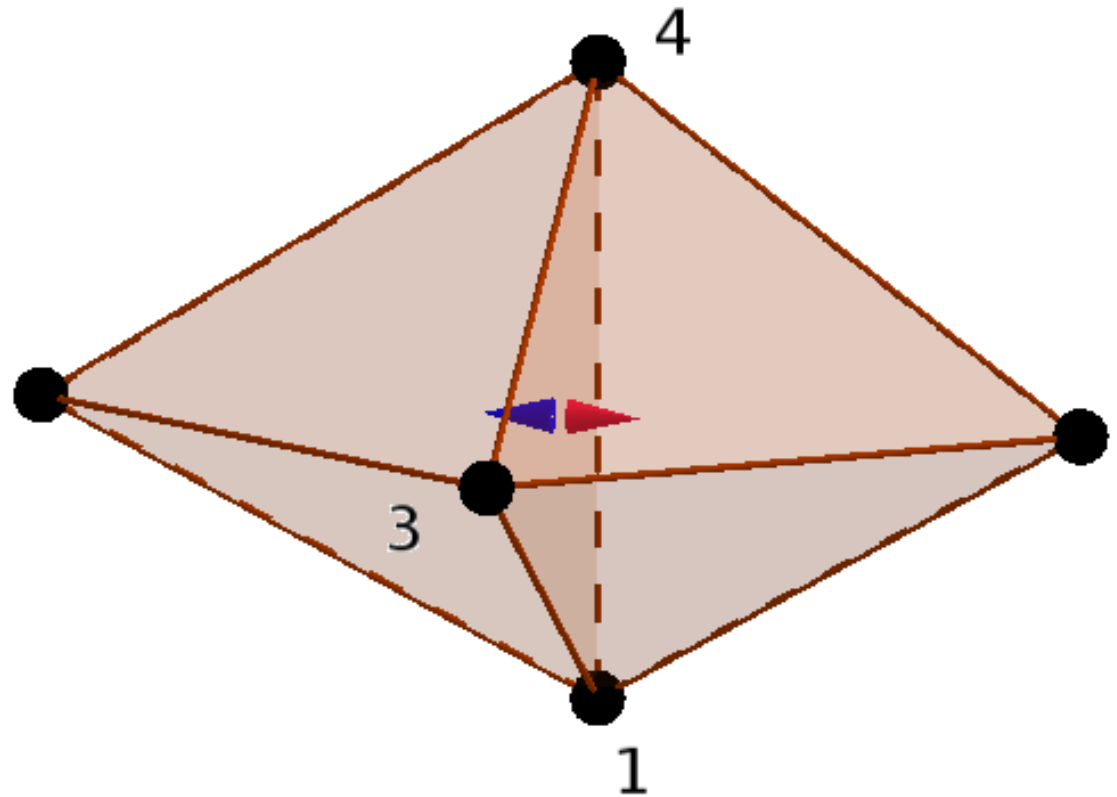
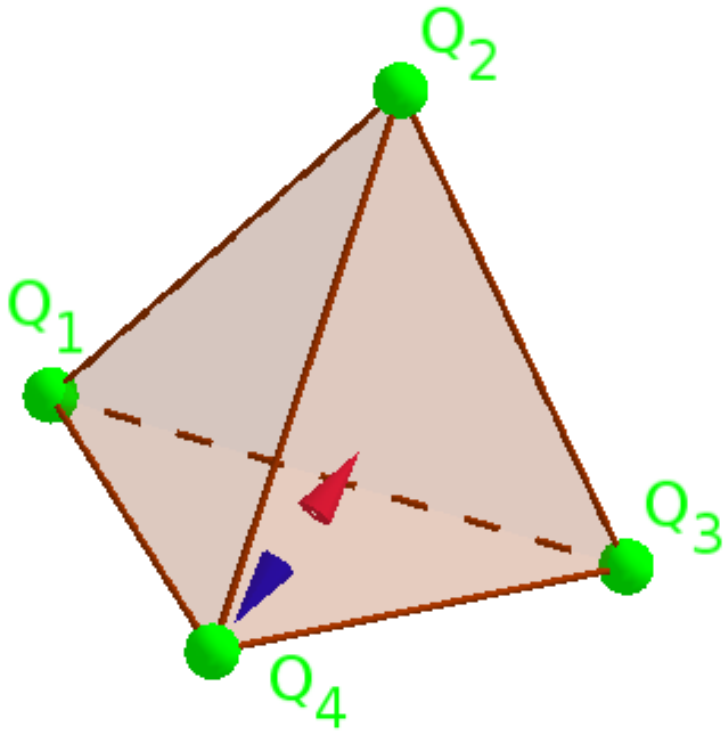
Degree of a labeling of an n -simplex in R^{n-1} ,
 = sum of degrees on each face oriented *inwards*:



Step 3: Boundary degree = Interior degree

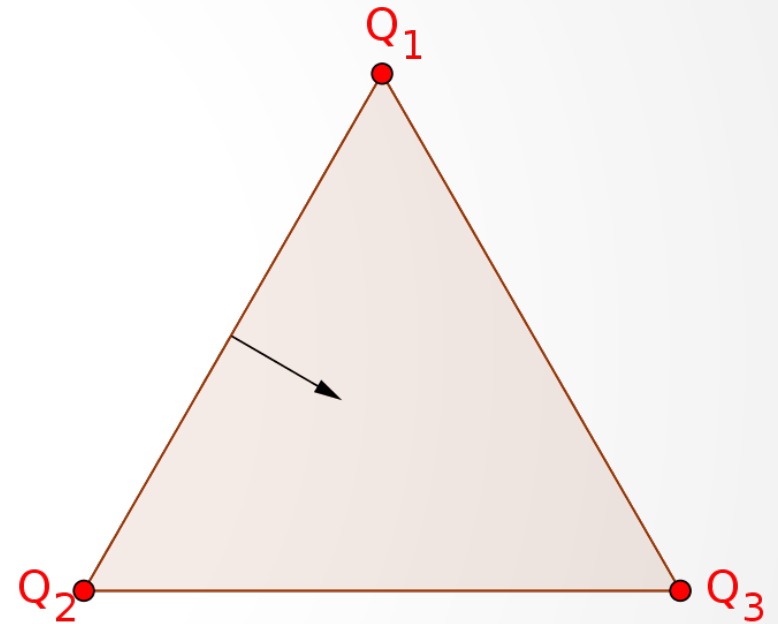
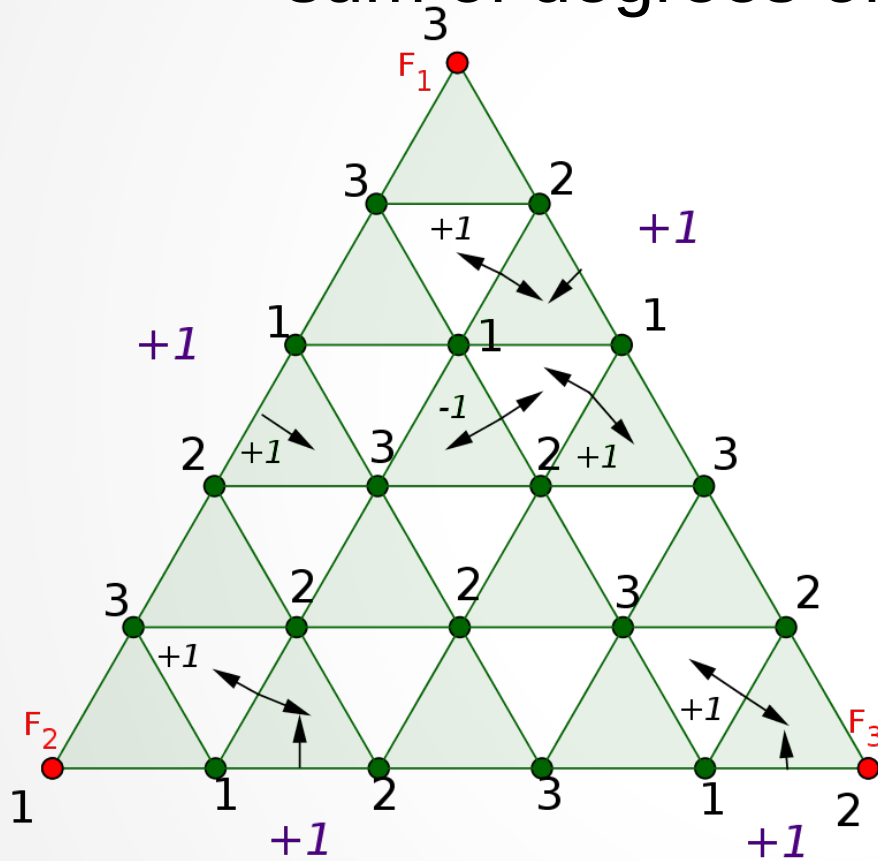
Lemma:

Sum of degrees of simplices in triangulation
= sum of degrees on each *boundary* face,
– since the internal faces cancel out:



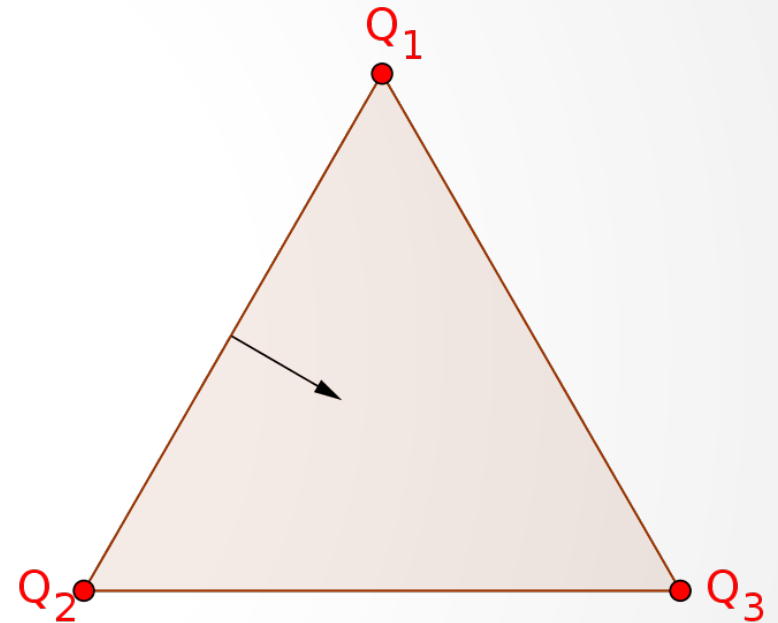
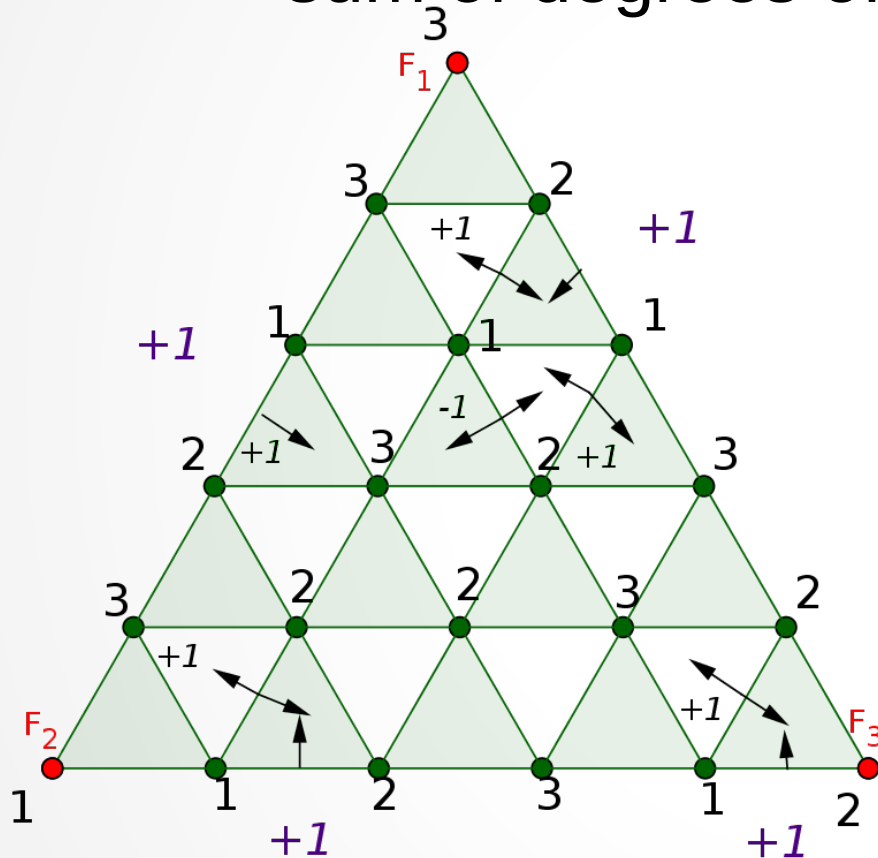
Step 3: Boundary degree = Interior degree

Definition: degree of triangulation labeling
 = sum of degrees of each sub-simplex labeling.



Step 3: Boundary degree = Interior degree

Definition: degree of triangulation labeling
 = sum of degrees of each sub-simplex labeling.



Lemma: interior degree = sum of degrees on faces
 = sum of degrees on faces of boundary = boundary degree.

Conclusion

Step	Proved for
1. n agent-labelings with perm. condition → Combined labeling with perm. condition	Any n
2. Permutation condition → Nonzero boundary degree	$n = 3$
3. Boundary degree = Interior degree	Any n (?)

Theorem: for 3 agents with continuous valuations, an envy-free connected division exists for arbitrary mixed valuations.

Open question

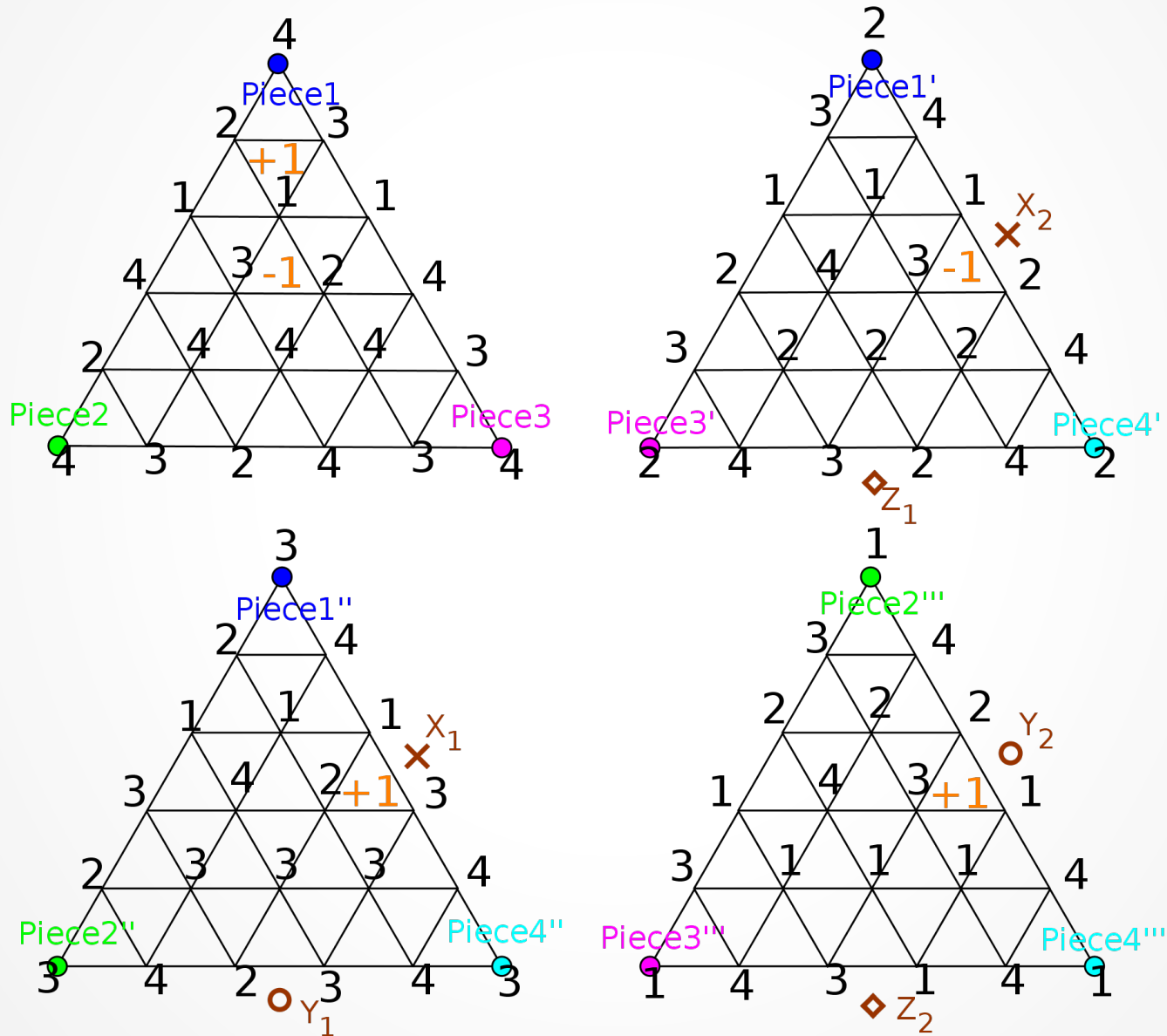
Permutation condition for 4 or more agents:

Pref:	Left	Middle	Right	Empty	
F_{123}	1	2	3	4	<i>Even</i>
F_{124}	1	2	4	3	<i>Odd</i>
F_{134}	1	3	4	2	<i>Even</i>
F_{234}	2	3	4	1	<i>Odd</i>

Conjecture: If labeling satisfies
permutation condition and agent condition,
then boundary-degree mod $n \neq 0$.

*If conjecture is true, then connected envy-free division exists
for arbitrary mixed valuations!*

Open question



Dividing Goods that are Bads

(Midrash Rabba, Genesis 33:1)



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