Fairly Dividing a Cake after Some Parts were Burnt in the Oven

Erel Segal-Halevi

Fair Division — Definition

Dividing a **heterogeneous** resource to agents with **different** preferences such that everyone's share is "fair" **by their preferences.**

Fair Division — Then

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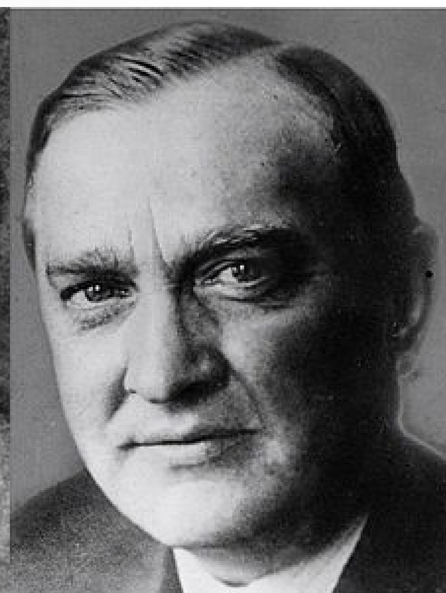
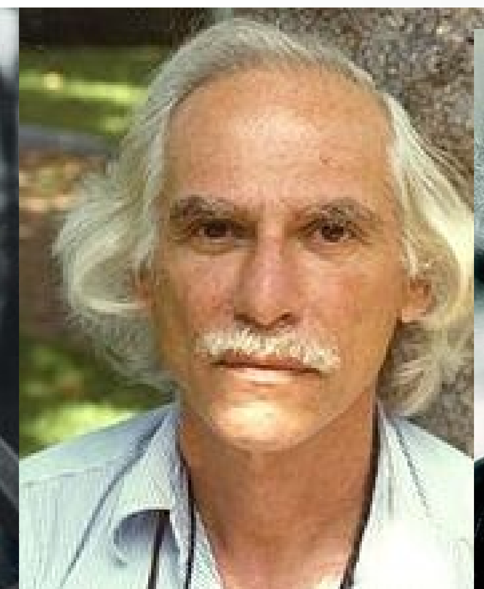
Steinhaus

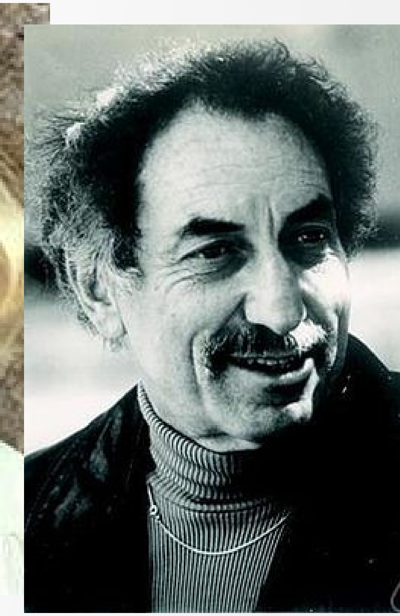


Banach

Knaster



Dubins



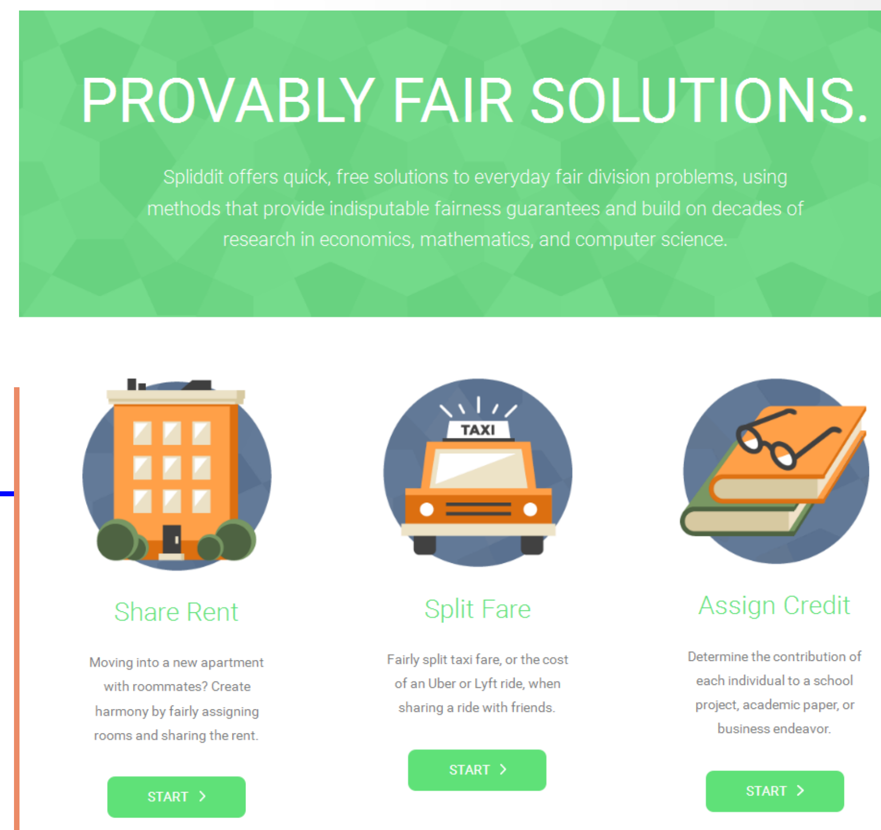
Spanier

Fair Division — Today

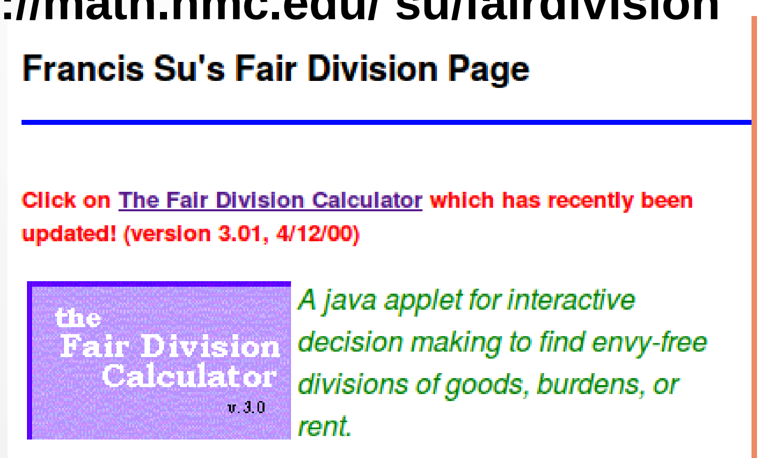
Dividing a **heterogeneous** resource to agents with **different** preferences such that everyone's share is "fair" **by their preferences.**

http://fairoutcomes.com

http://spliddit.org



https://math.hmc.edu/~su/fairdivision



Fair Division — Examples

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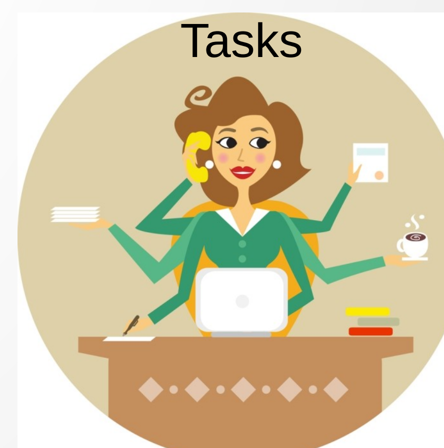

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Continuous Resource

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n agents. Value-densities $v_i : Cake \rightarrow \mathbf{R}$

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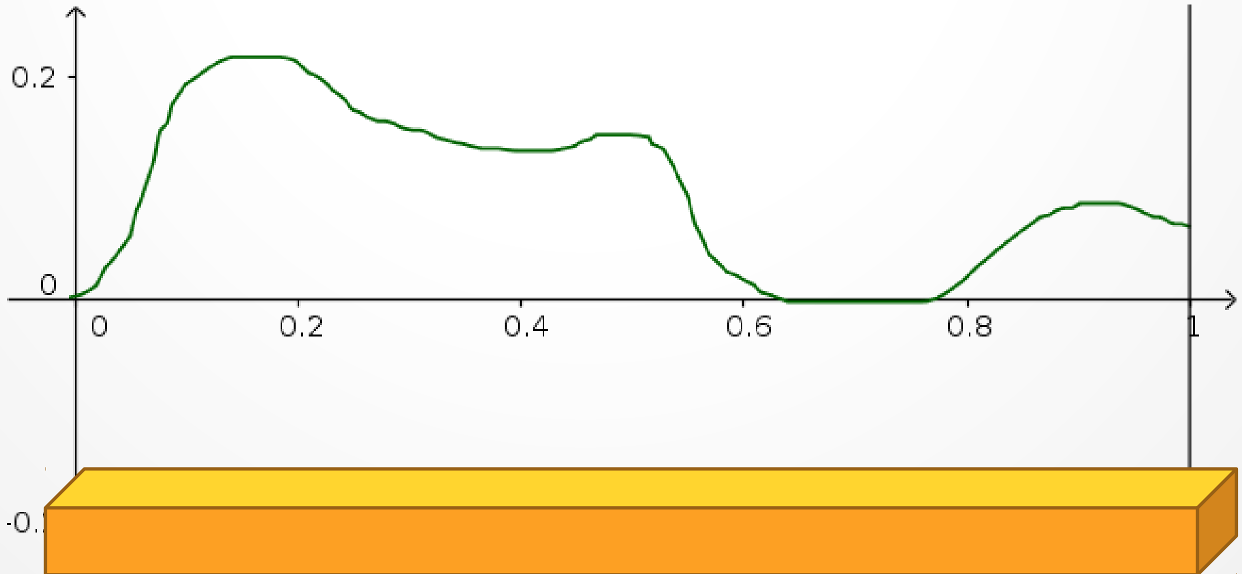

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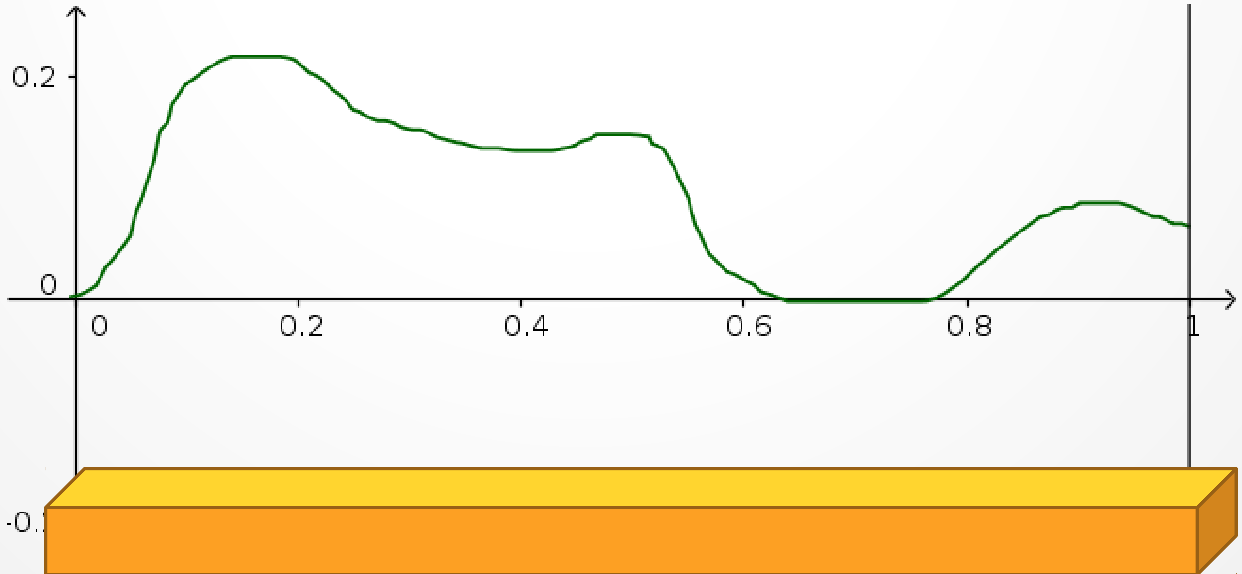
For all i: X_i is **connected.**



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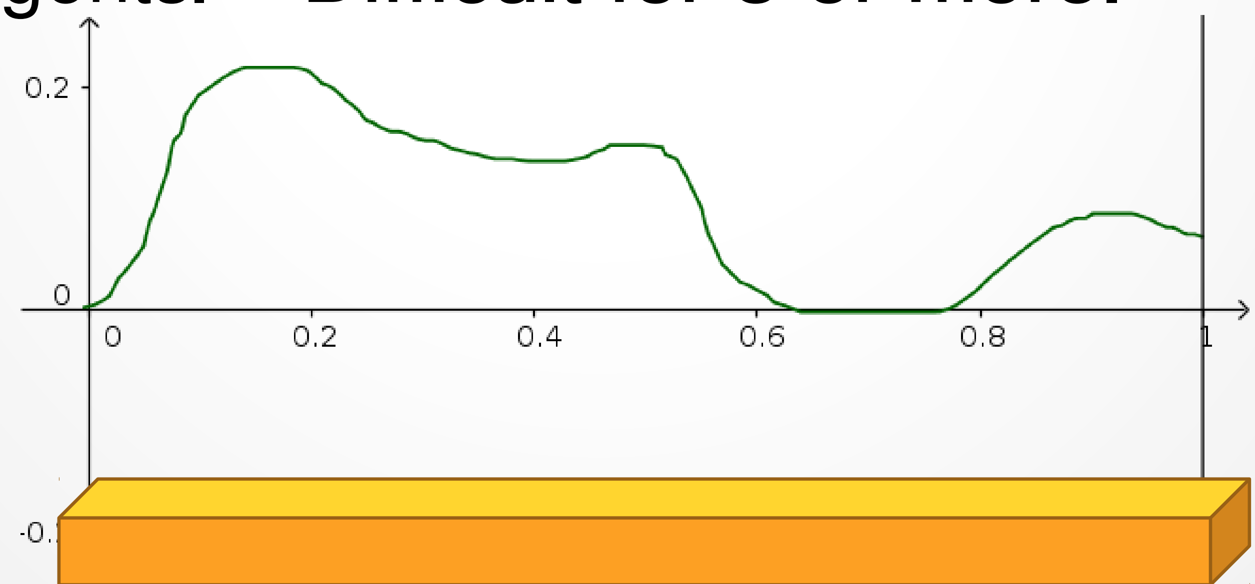
Easy for 2 agents. **Difficult for 3 or more.**



Valuation types

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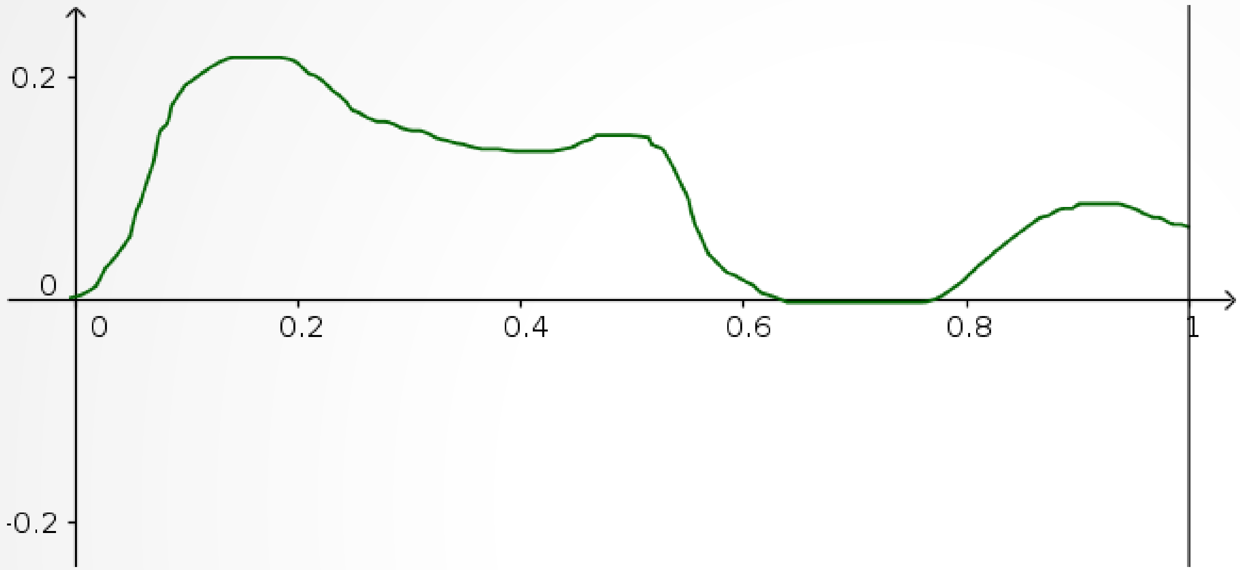

All positive - solved by Stromquist (1980), Simmons (1980)

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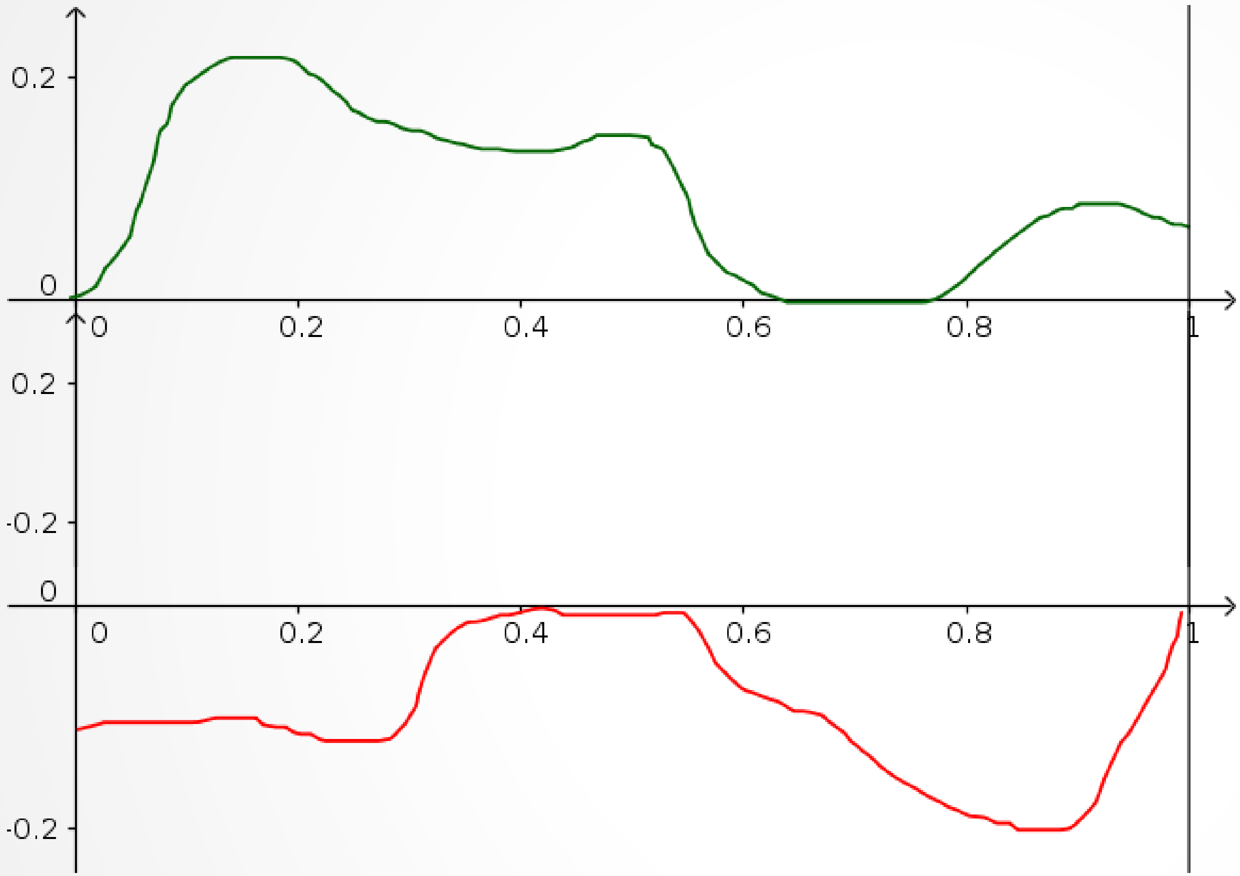

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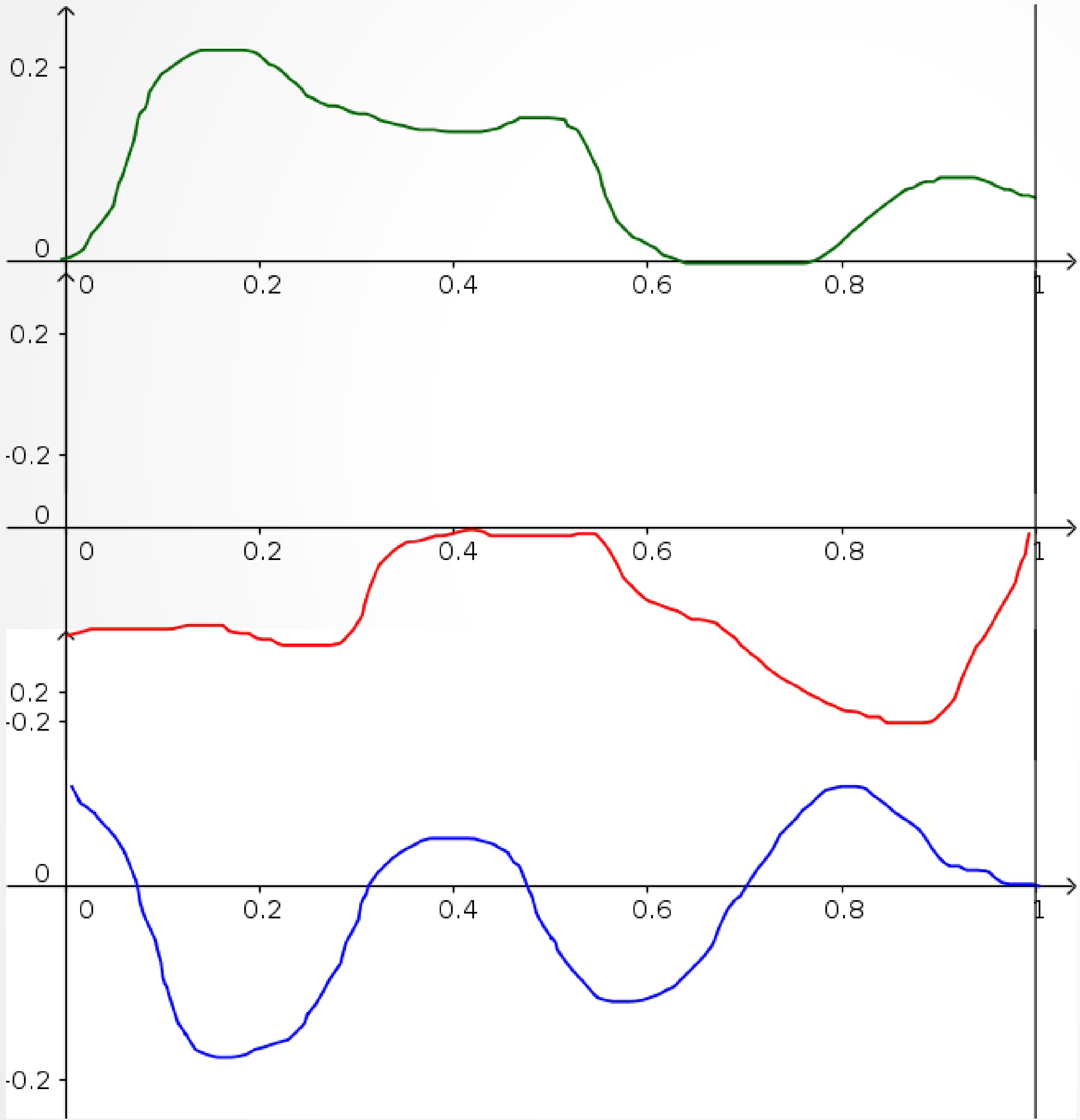

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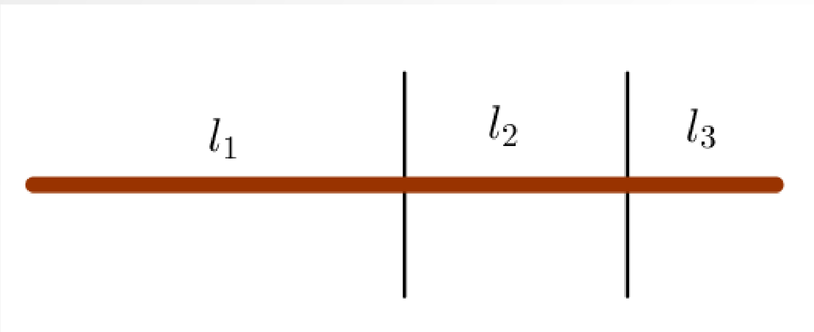
General – **this work.**

Simplex of Partitions – Definition
(based on Stromquist 1980)



Partition for 3 agents:

(l_1, l_2, l_3)

$l_1 + l_2 + l_3 = 1$

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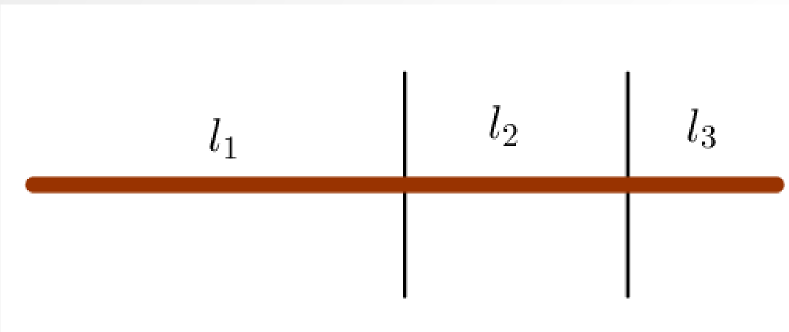

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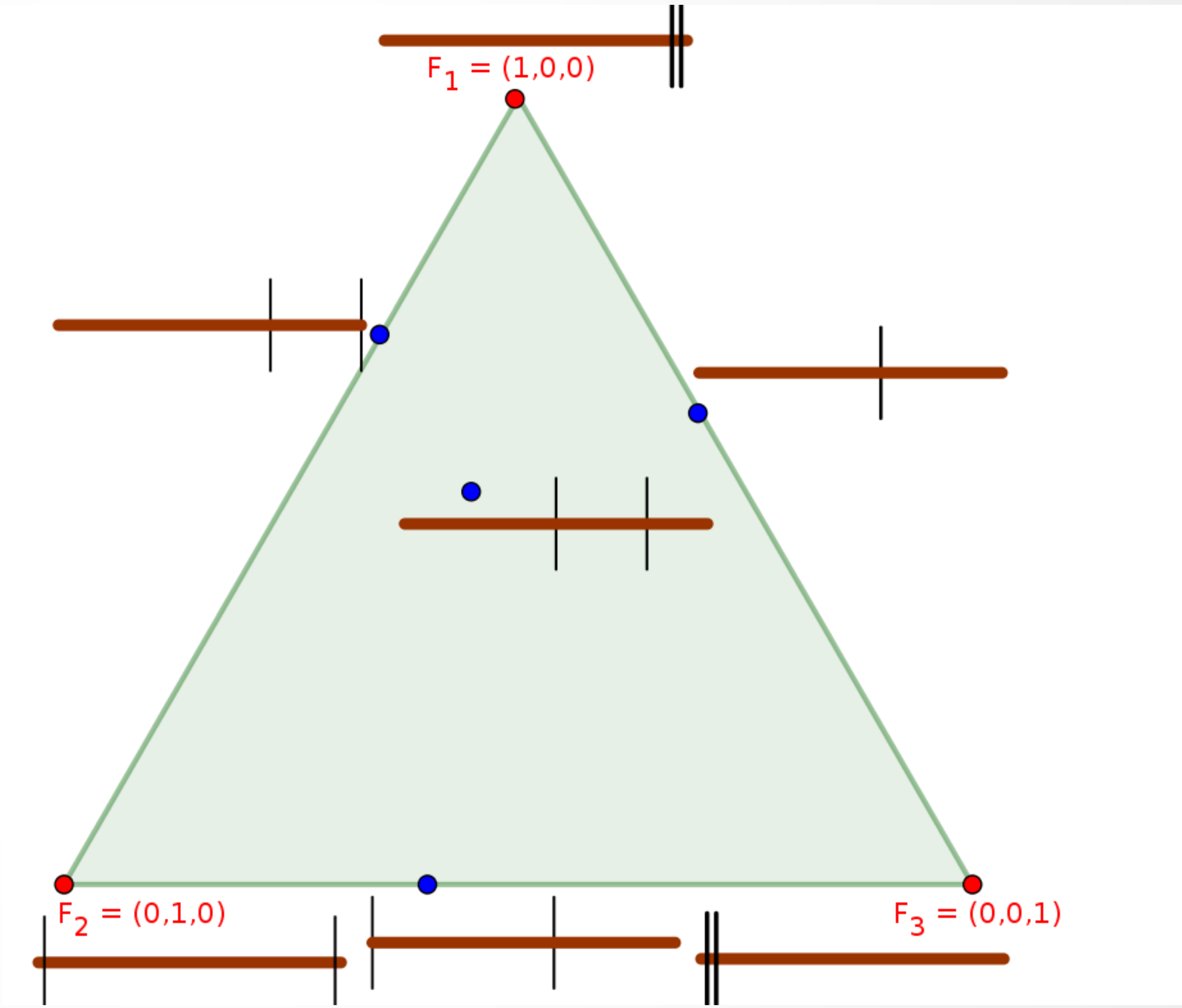

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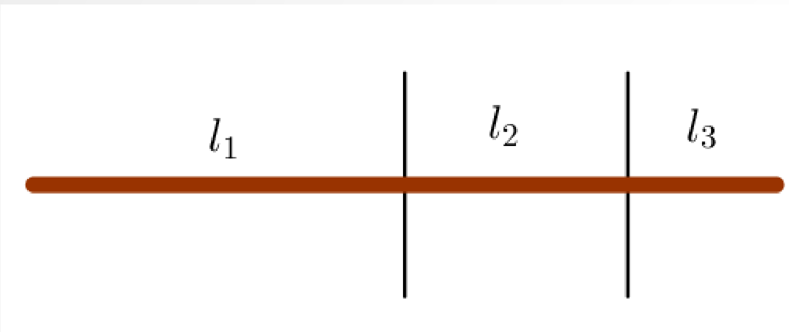

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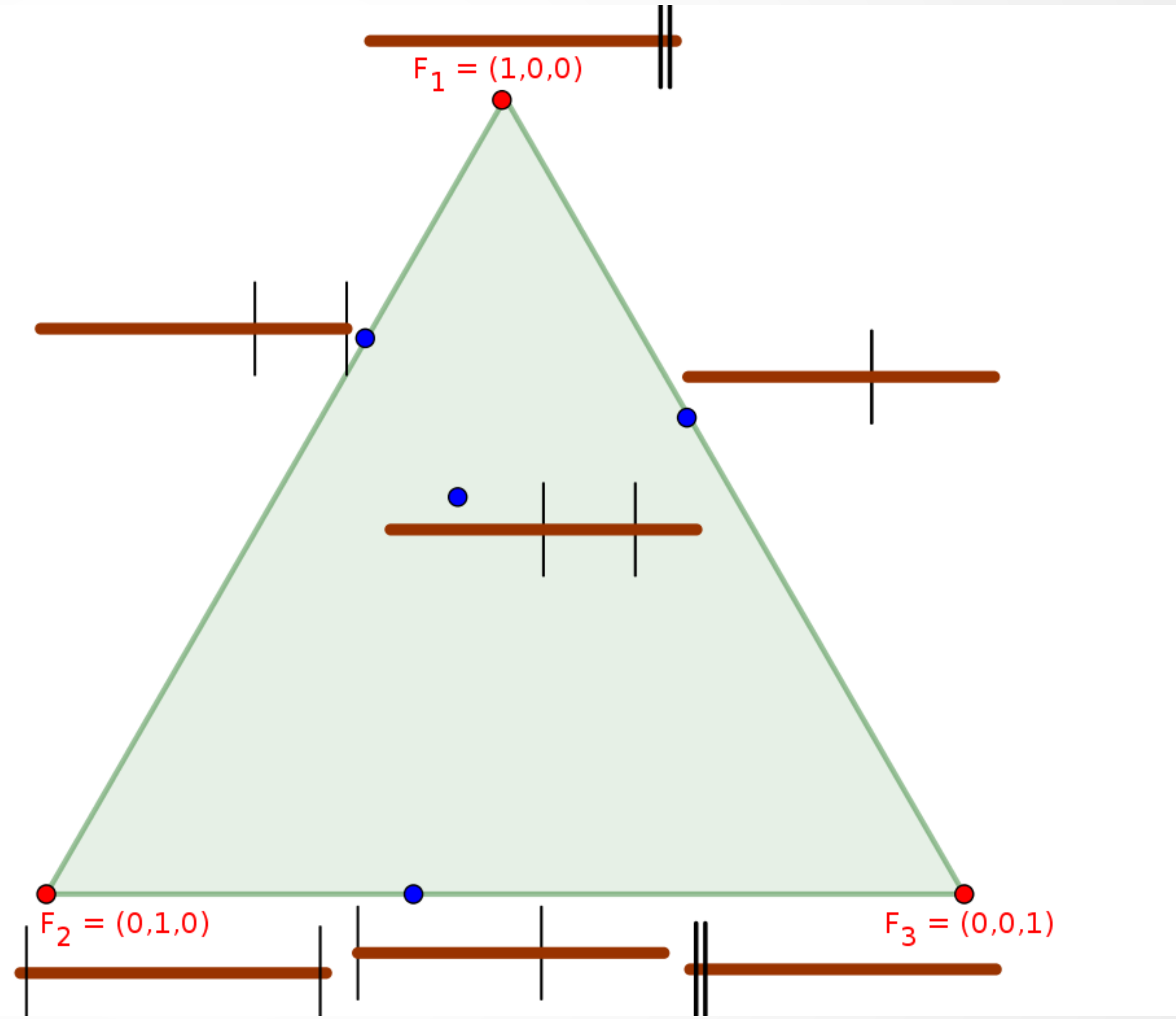
Envy-free division =



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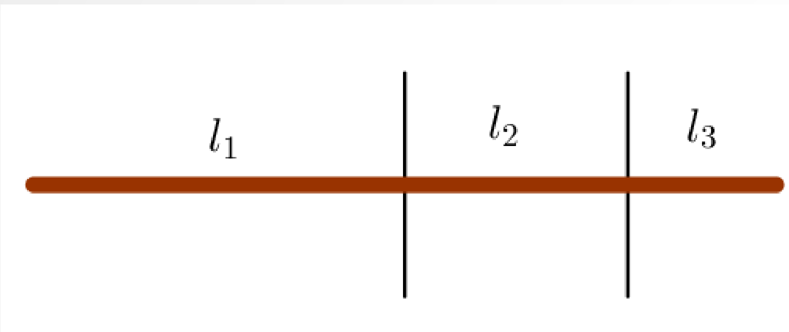

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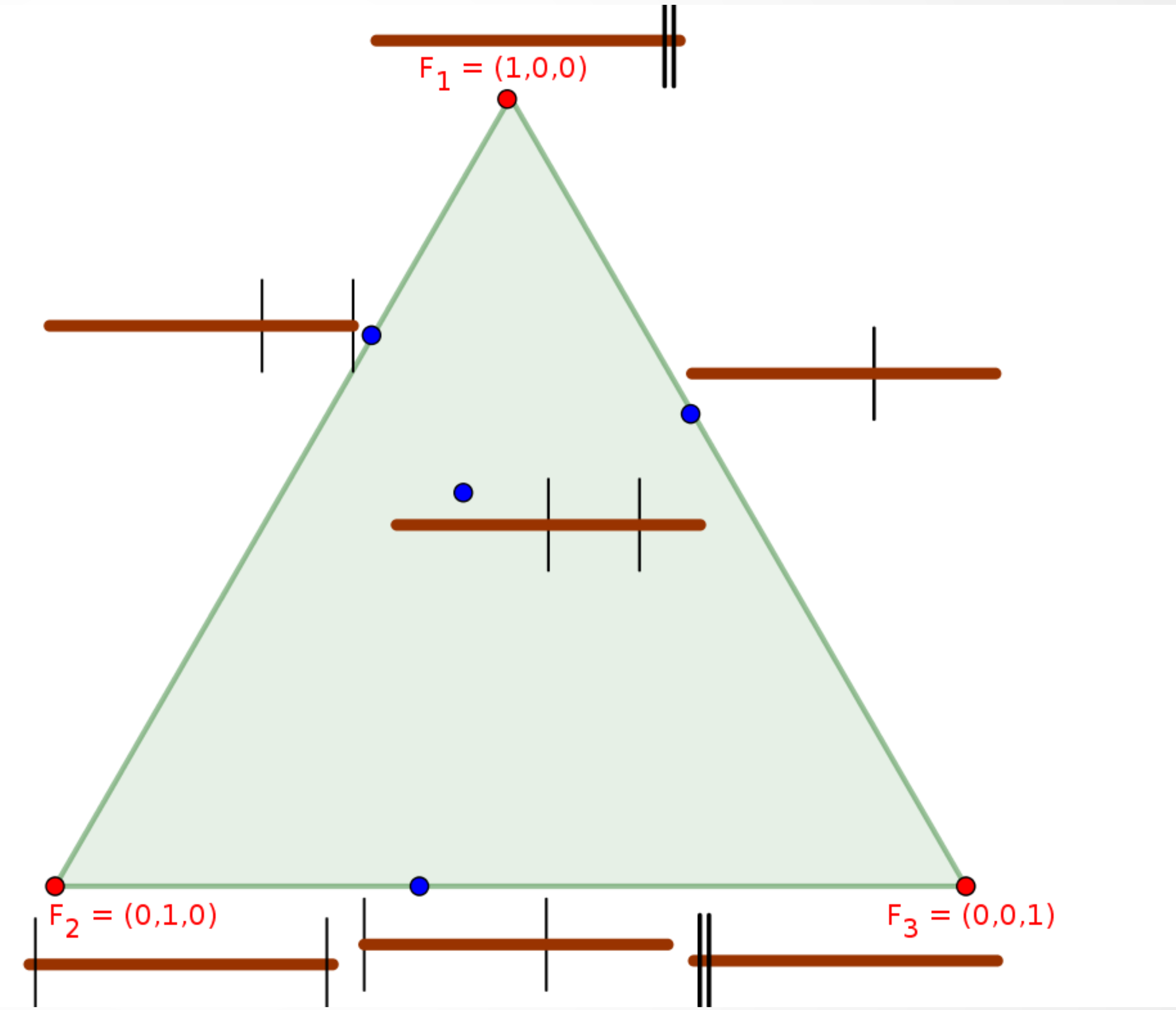
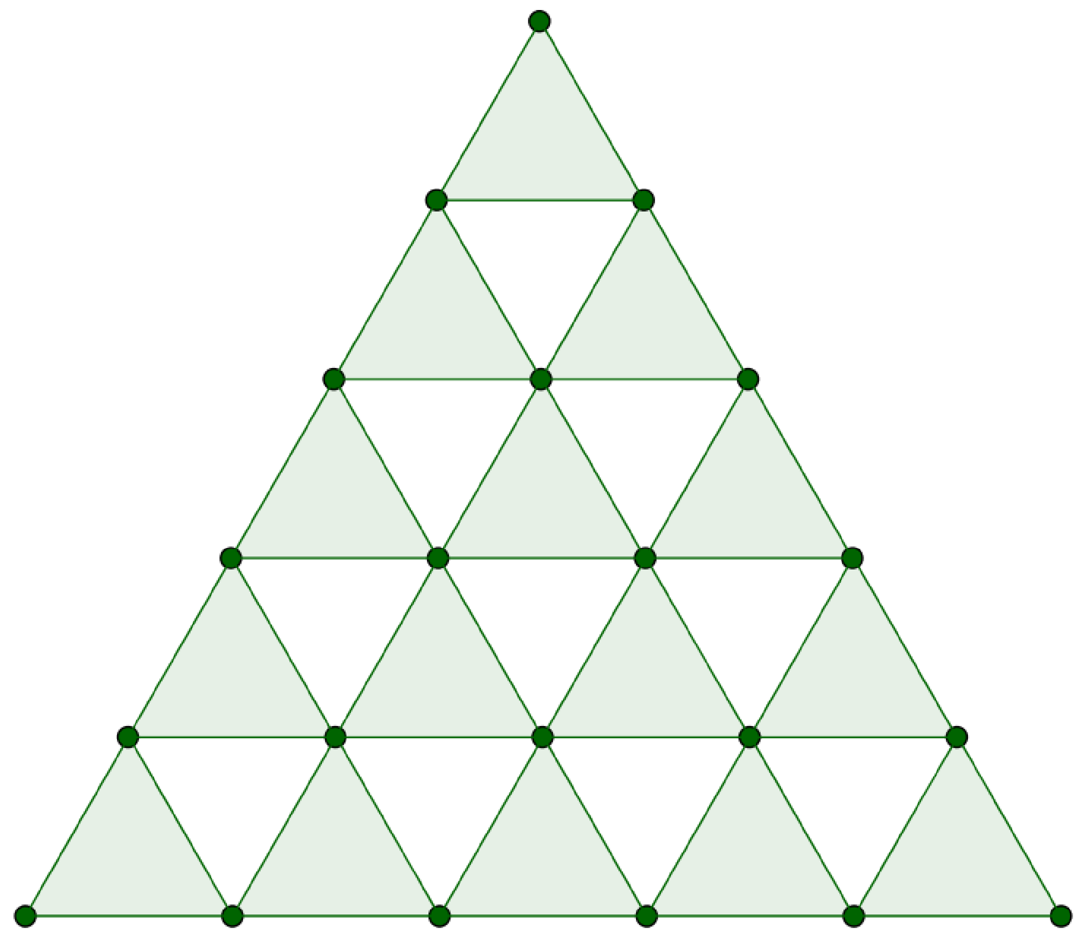
Envy-free division = point in which each agent prefers a different piece.



Simplex of Partitions – Triangulation
(based on Simmons 1980, Su 1999)

a. Triangulate the simplex.



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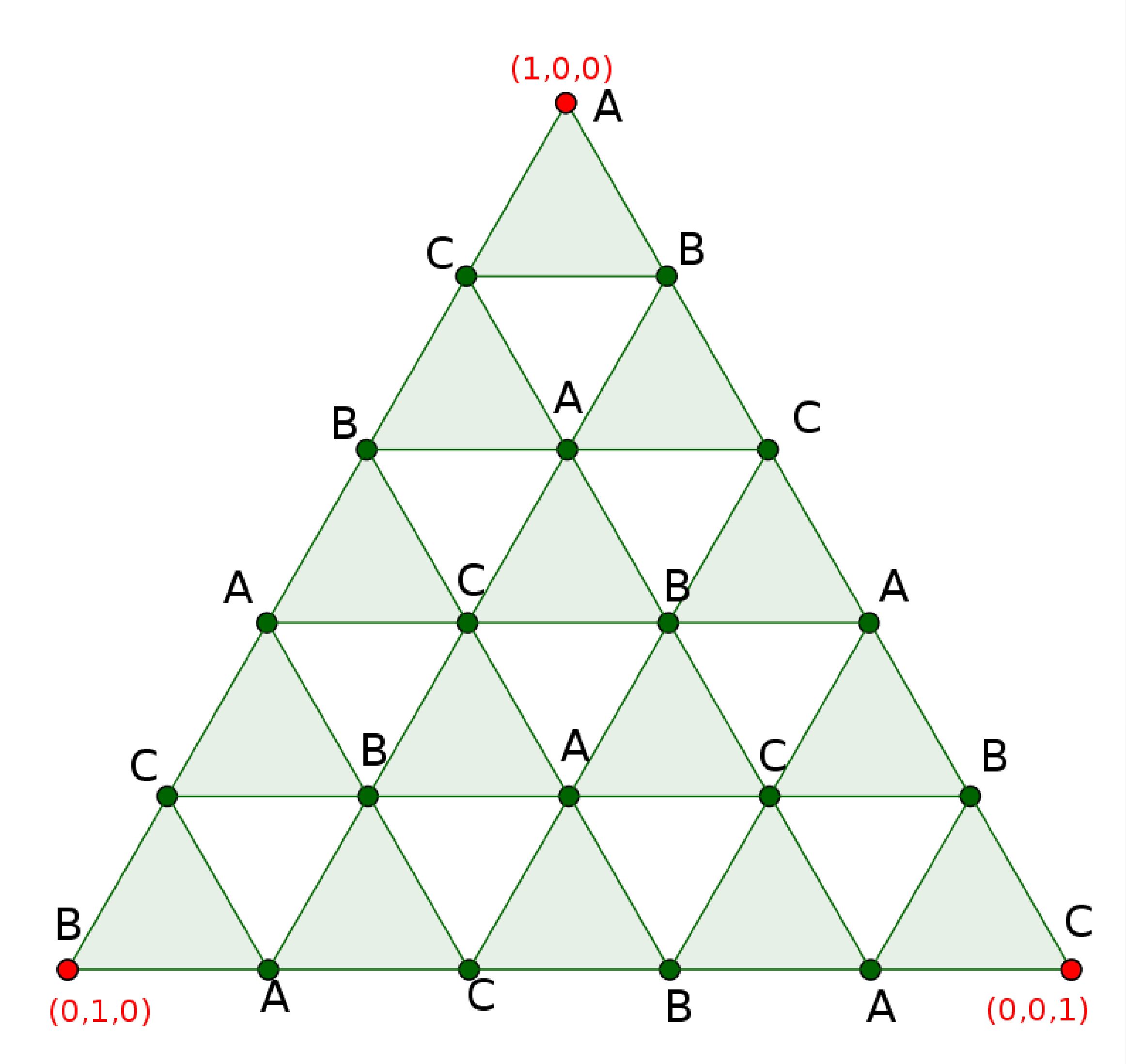
a. Triangulate the simplex.

b. Assign each vertex to a different agent such that in each sub-simplex, all agents are represented.



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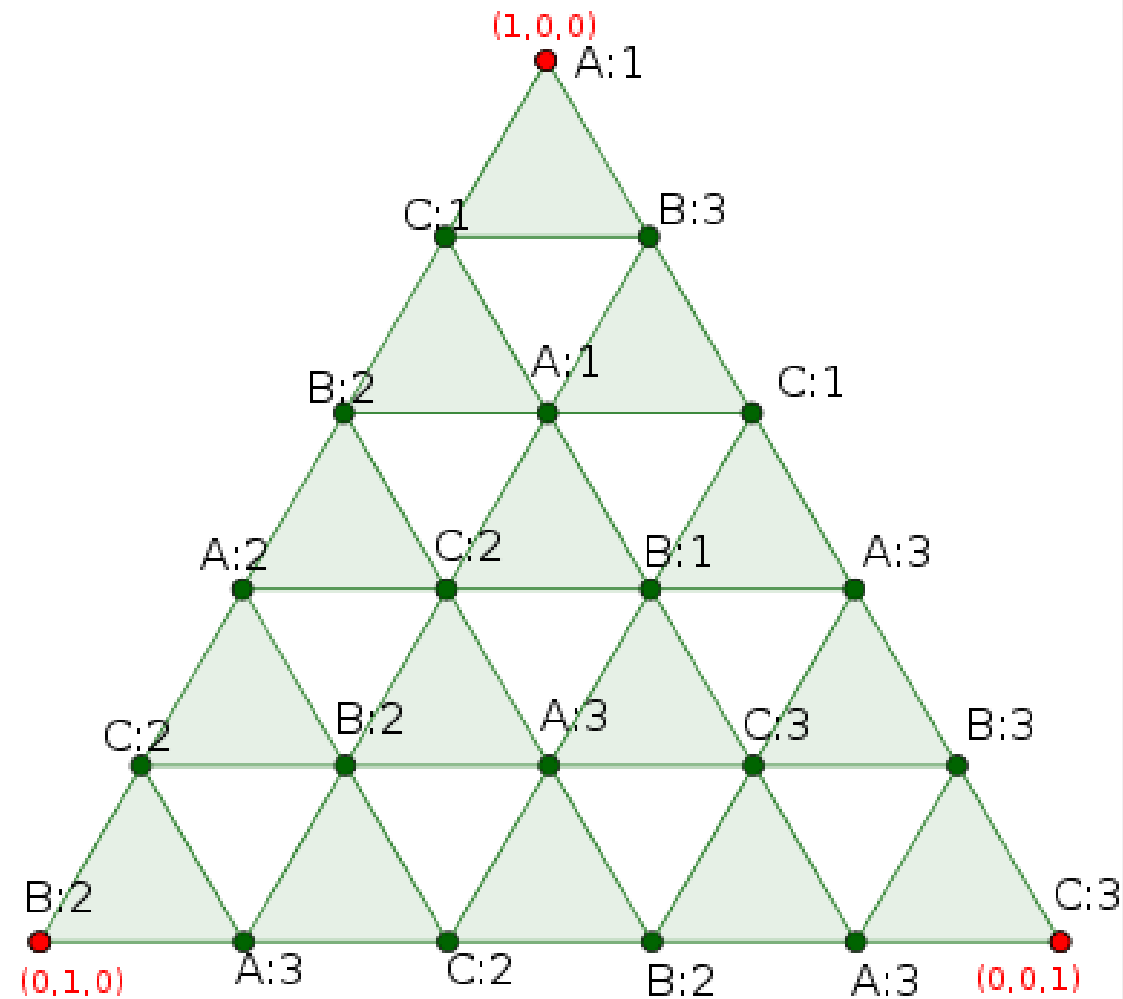

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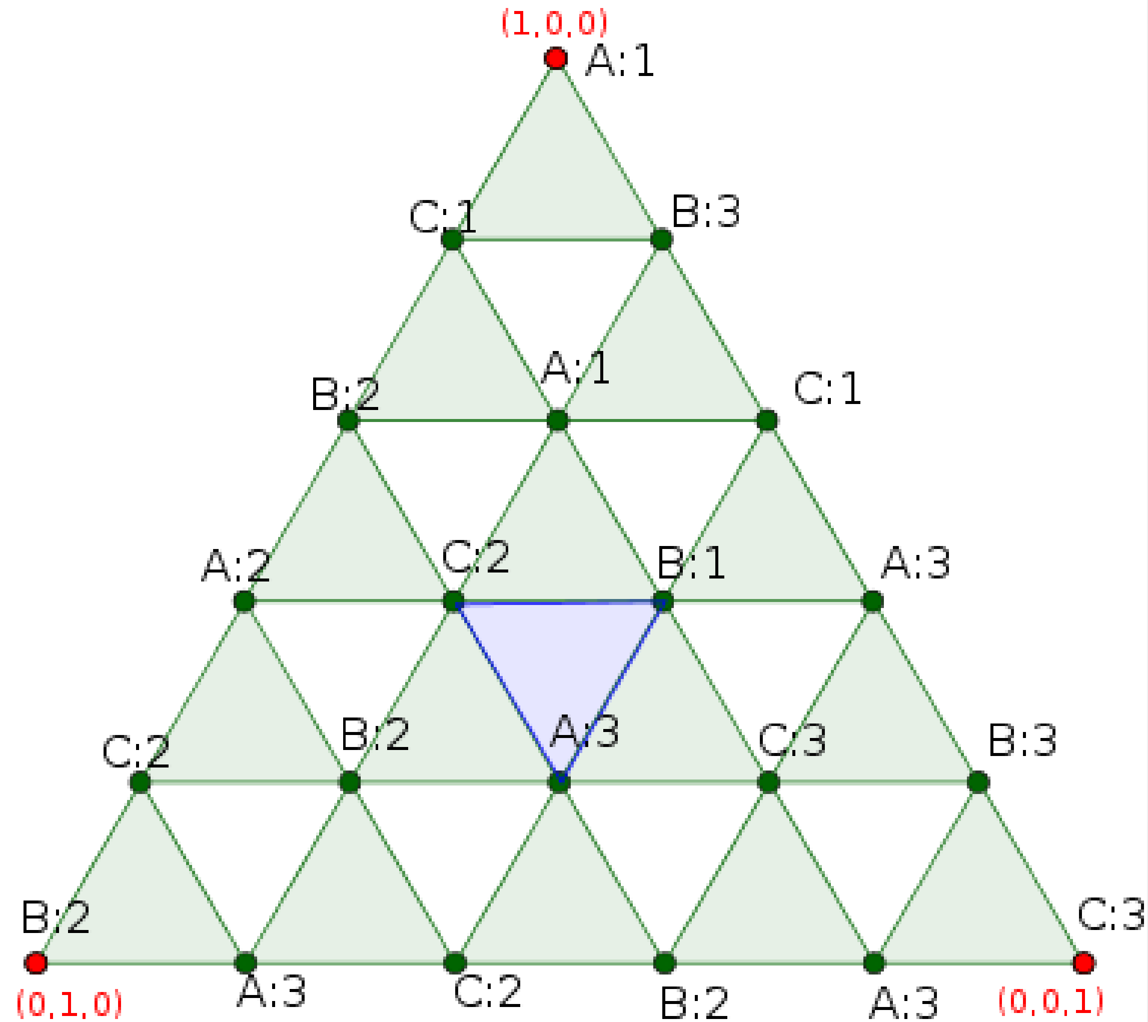
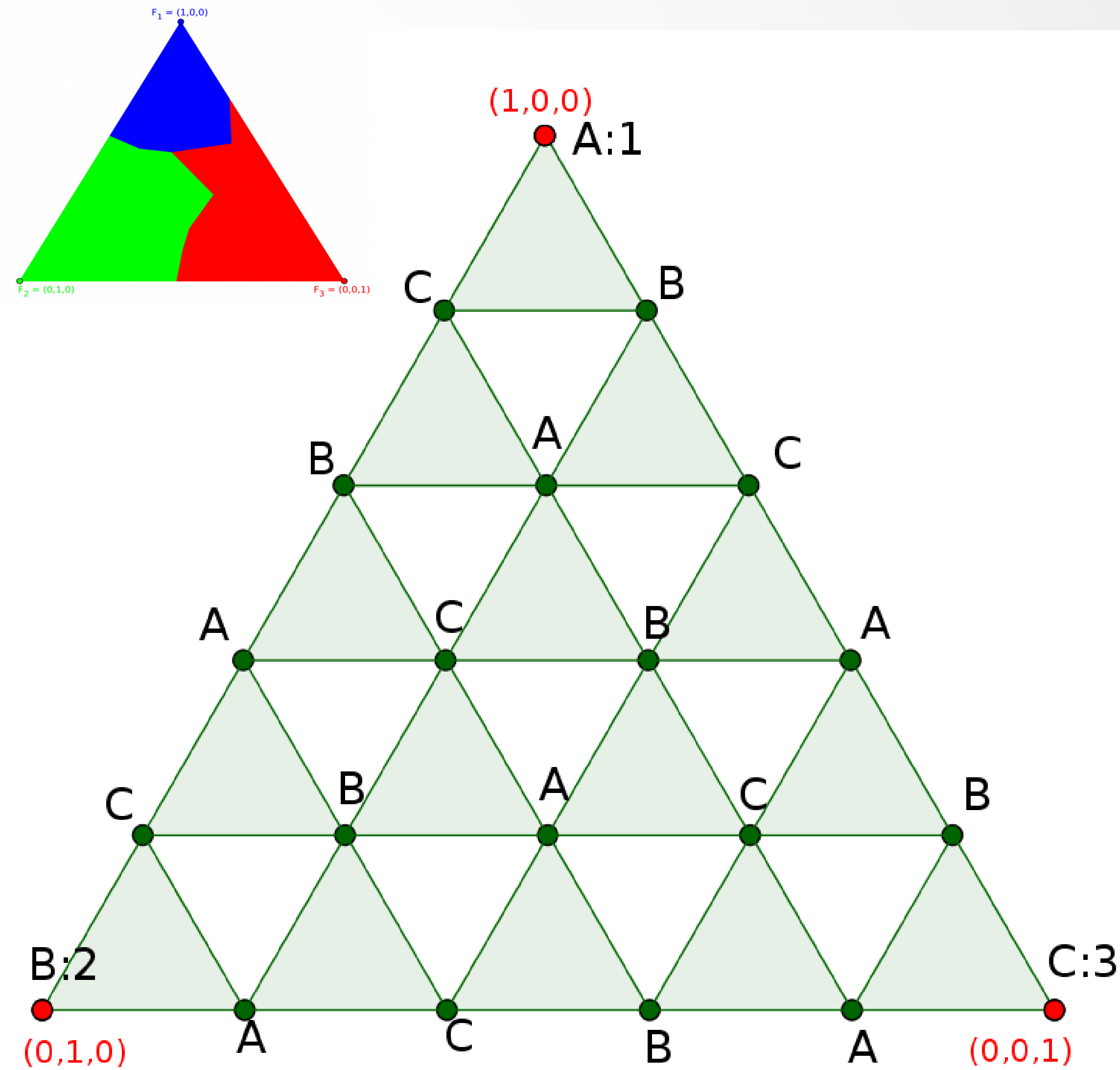
c. Ask each agent to label all its vertices by the index of his favorite piece.

d. A simplex labeled by all n labels = an approximately-envy-free division.



Triangulation – Positive Agents

Fact: When all agents have positive valuations, each face is labeled only with the labels of its endpoints.



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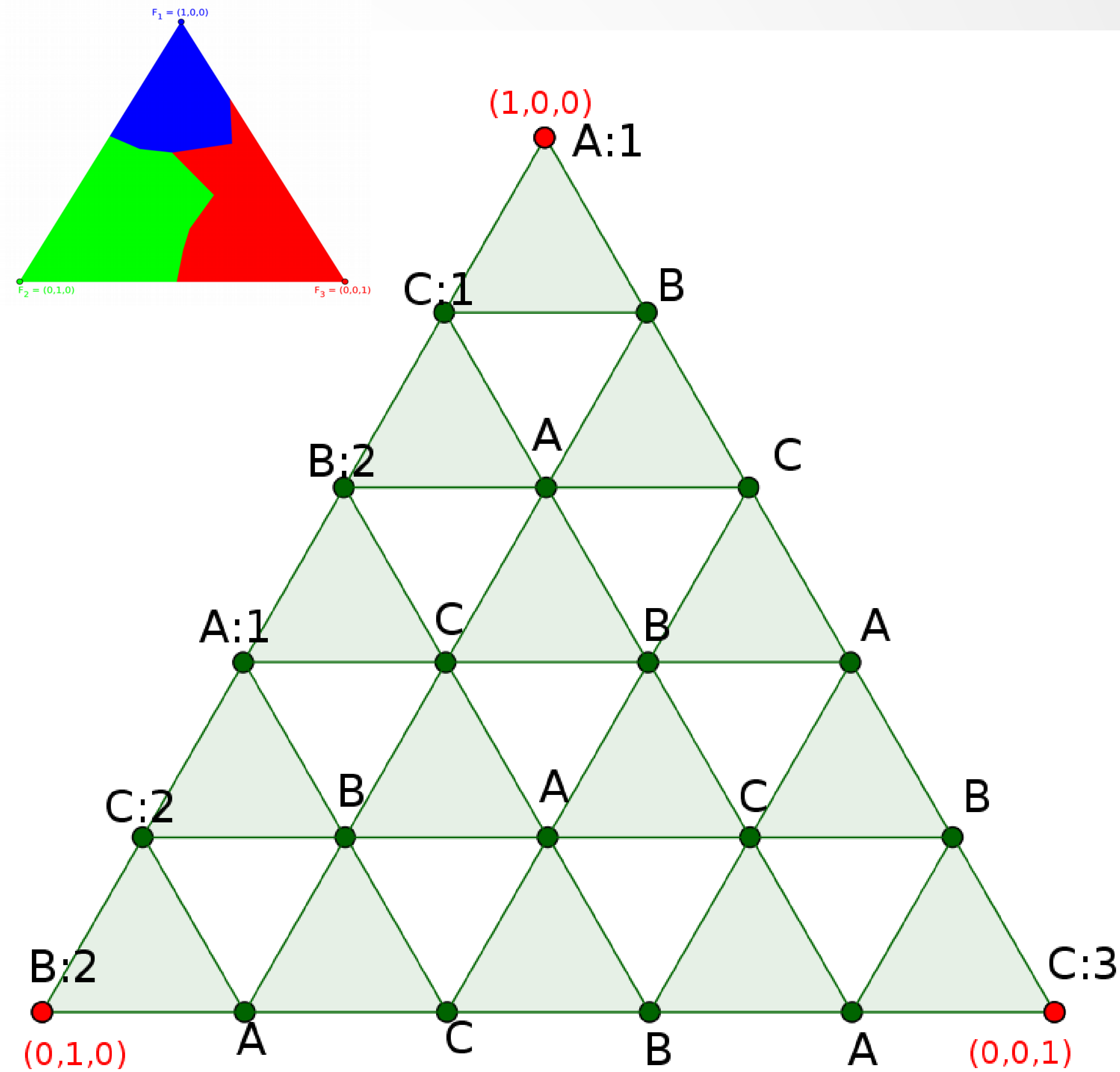

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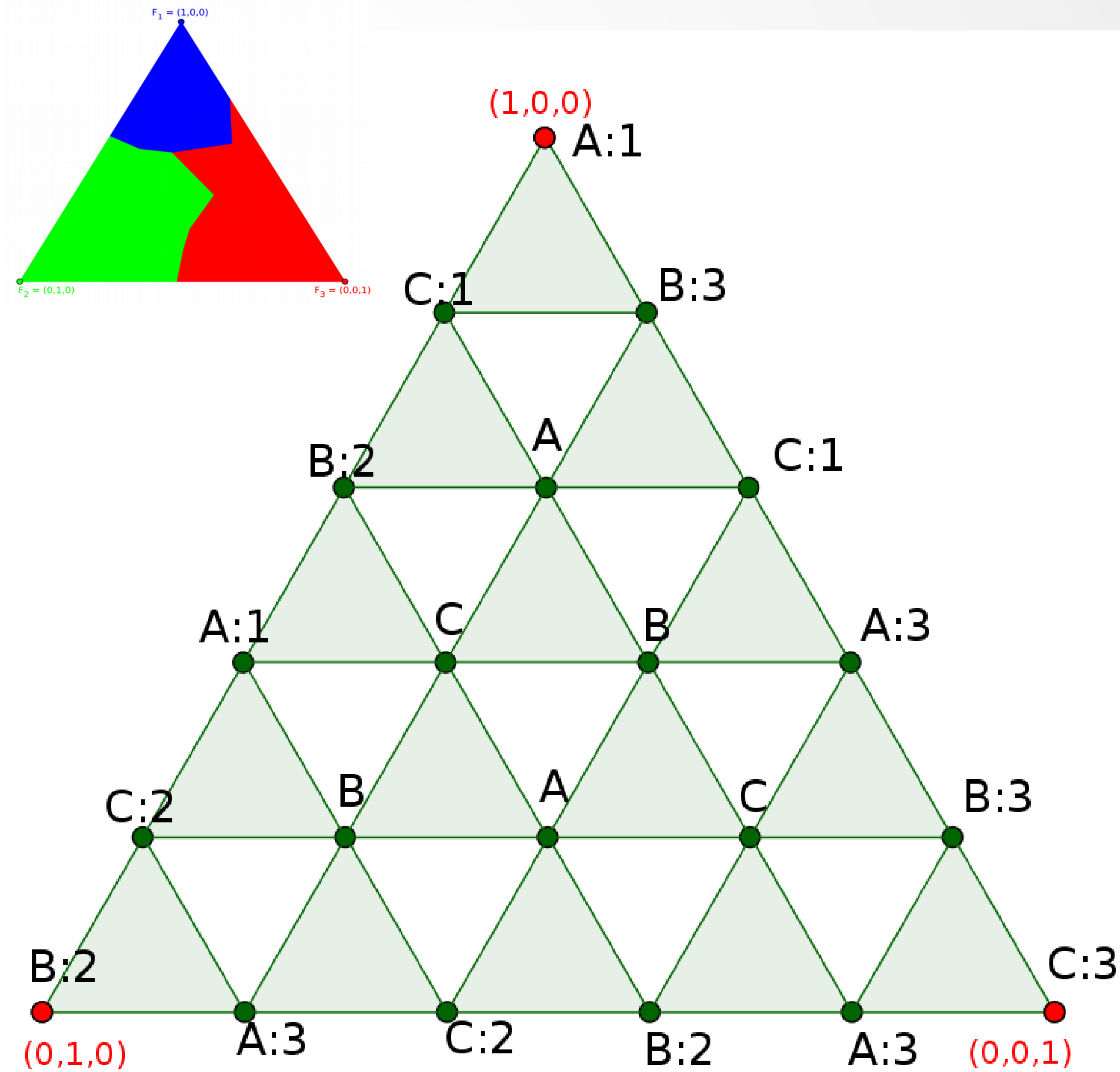

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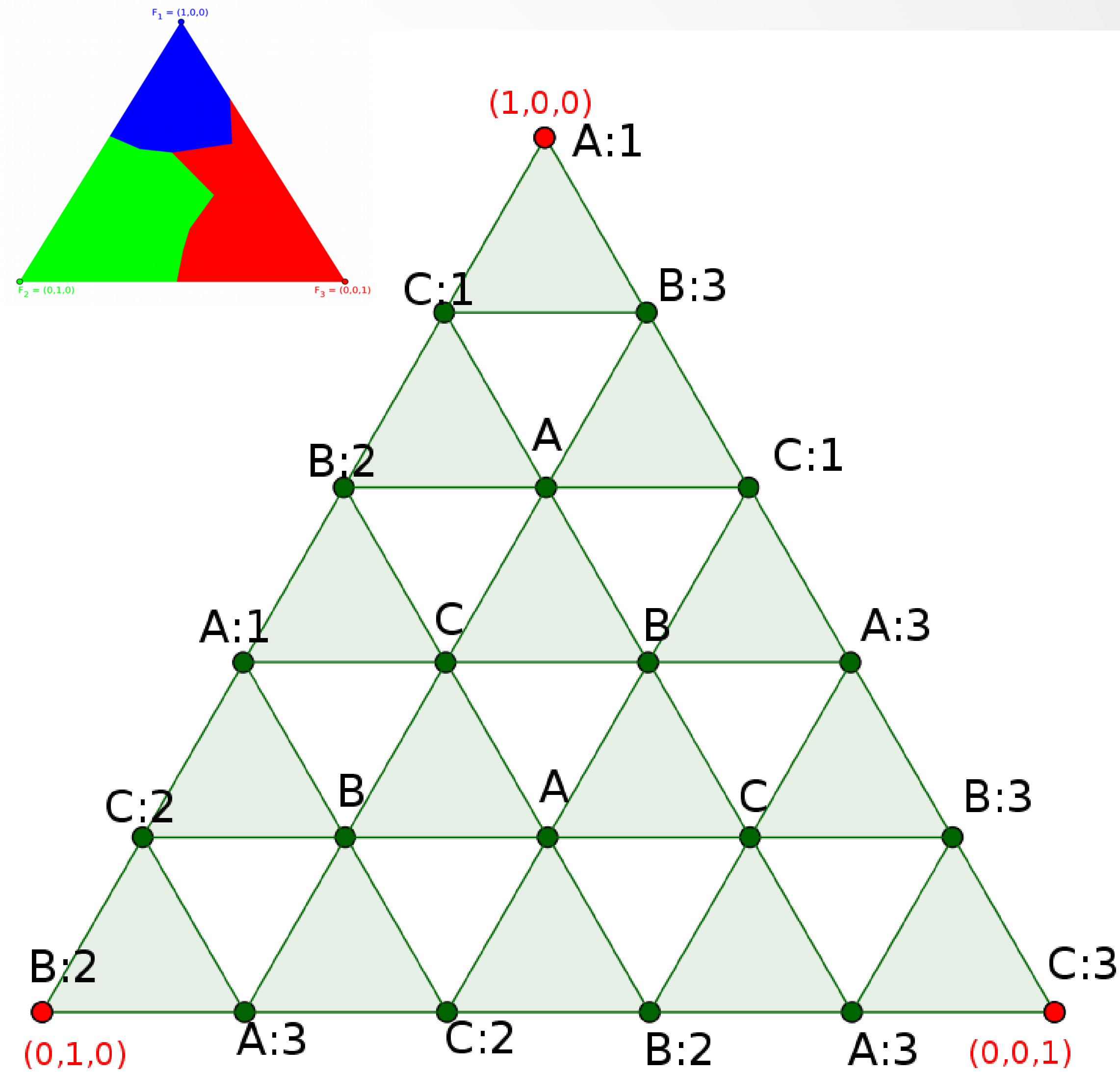
Lemma (Sperner 1929): When each face is labeled only with the labels of its endpoints, a fully-labeled sub-simplex exists.



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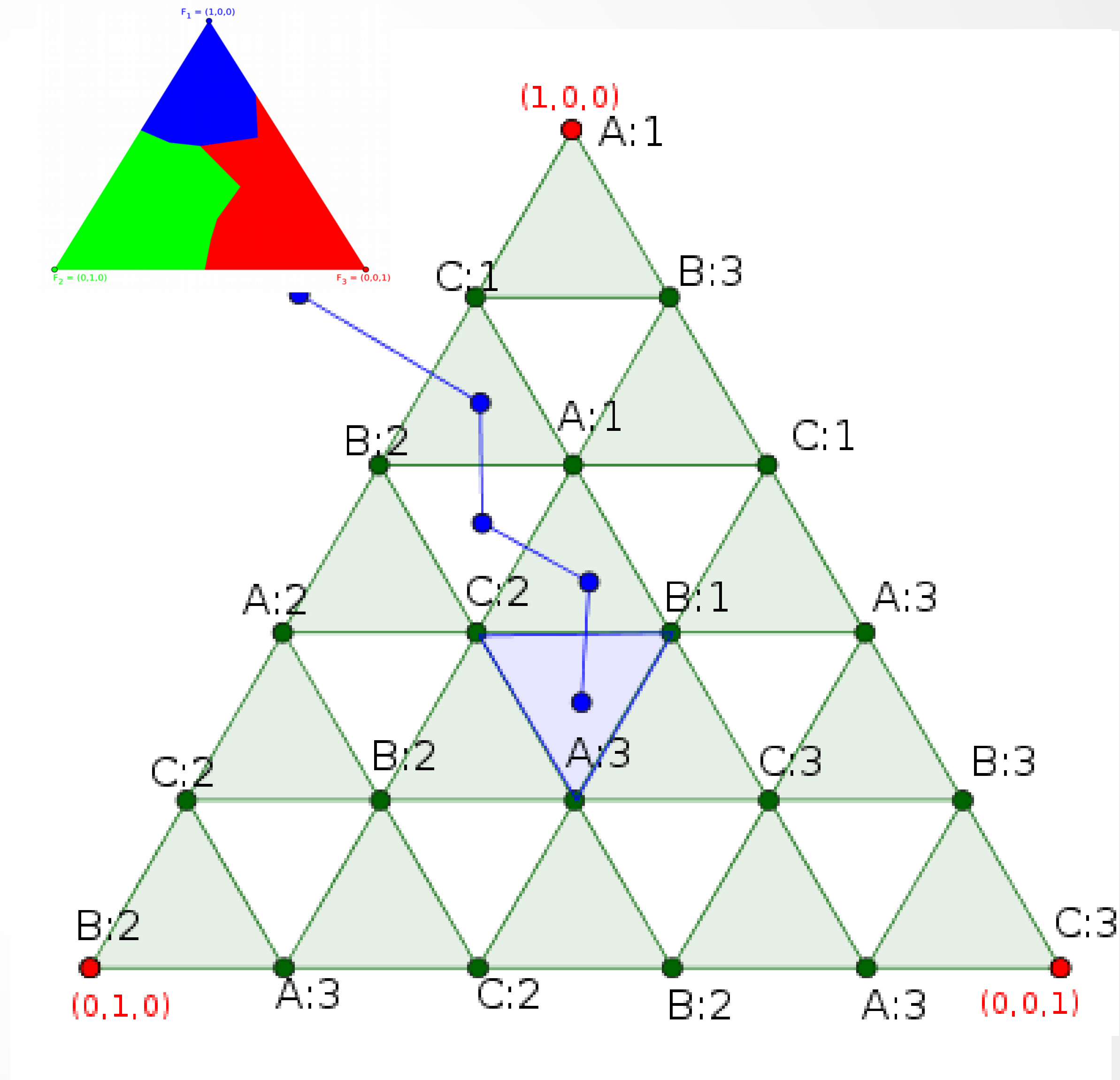

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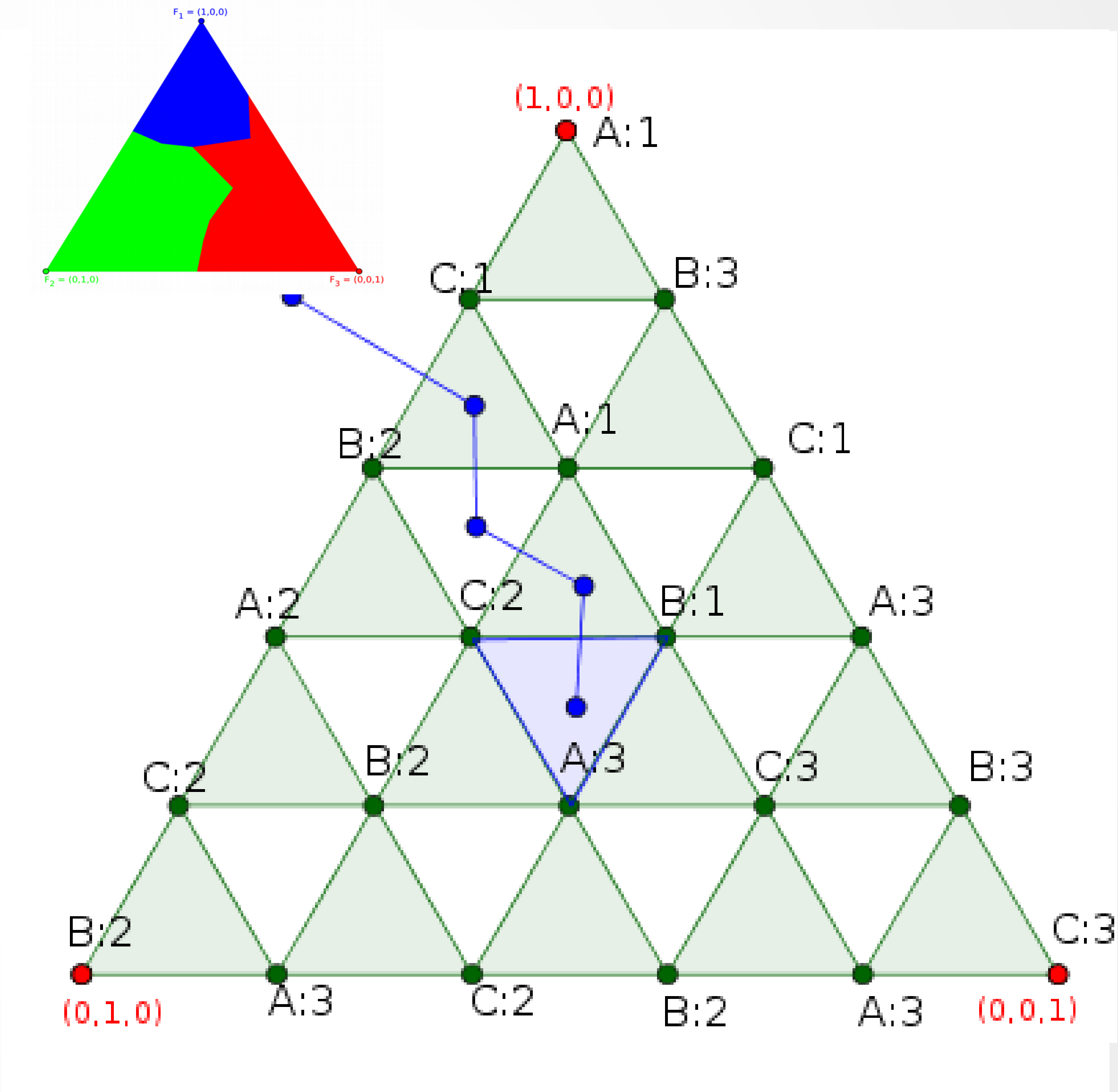
Corollary: when all valuations are positive, an approximately-envy-free division exists.



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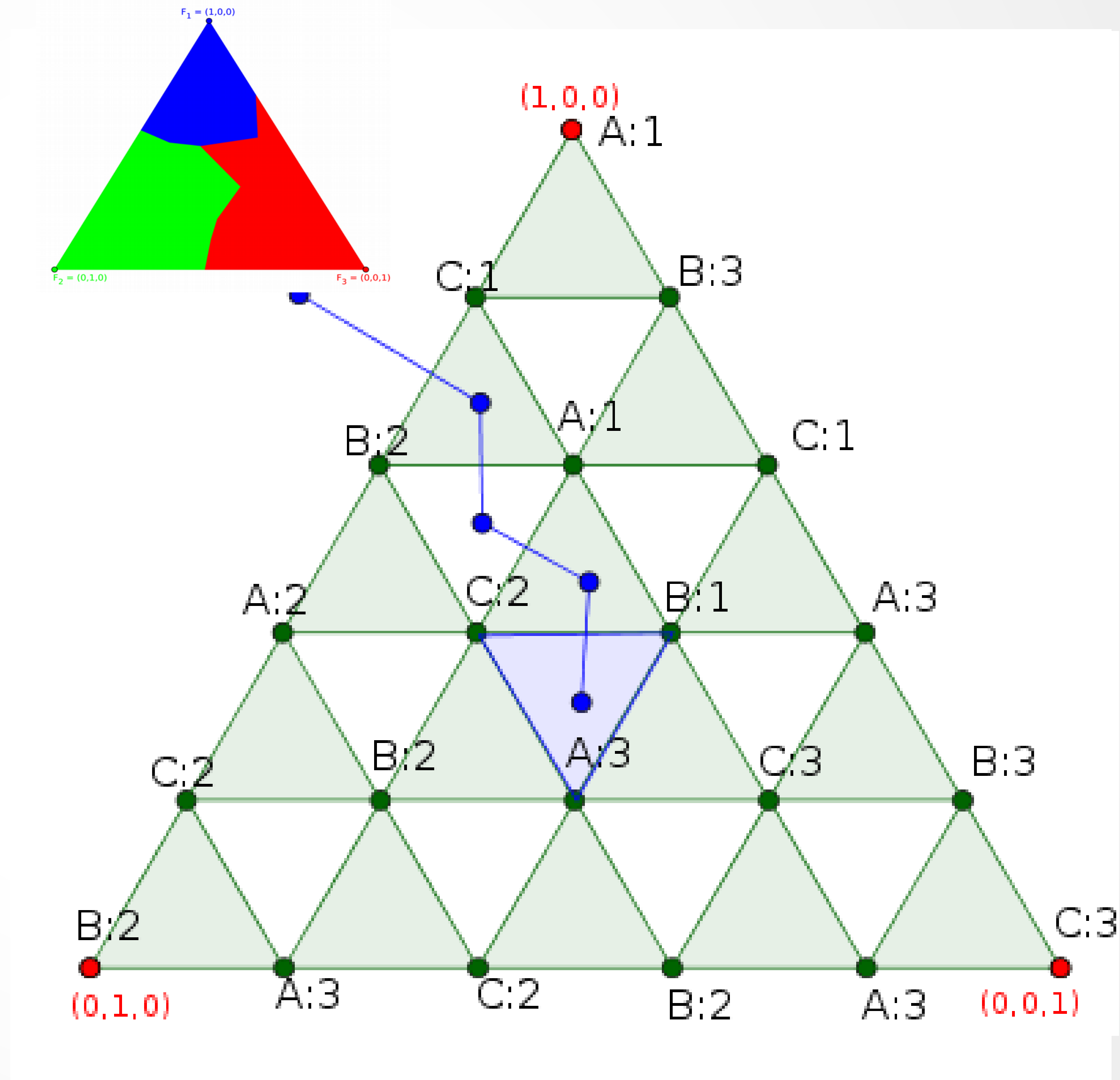
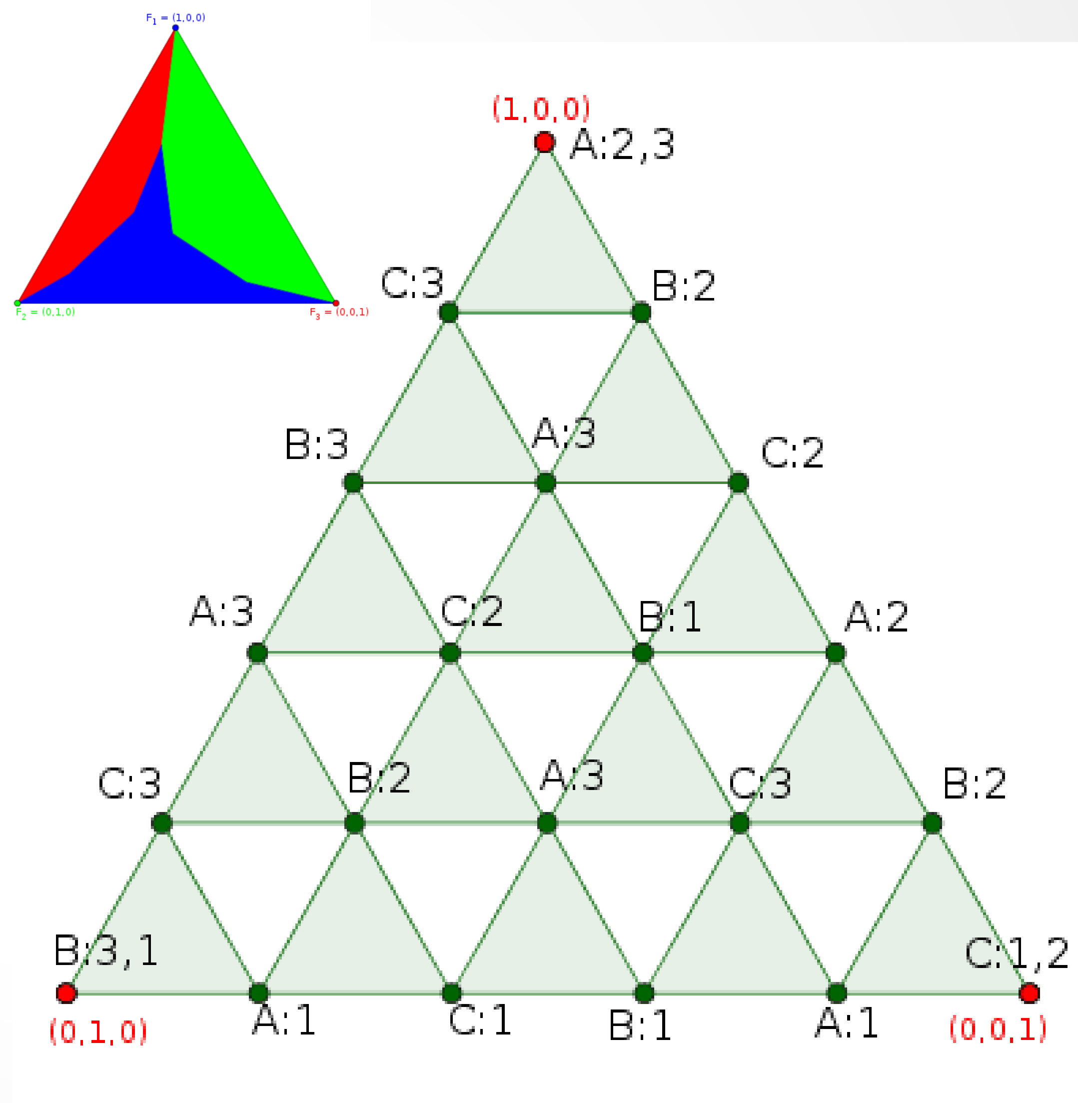
Corollary (Stromquist 1980, Simmons 1980, Su 1999): when valuations are also continuous, an envy-free division exists.



Triangulation – Negative Agents

Fact: When all agents have negative valuations, it is possible label the n main vertices such that each face is labeled only with the labels of its endpoints.



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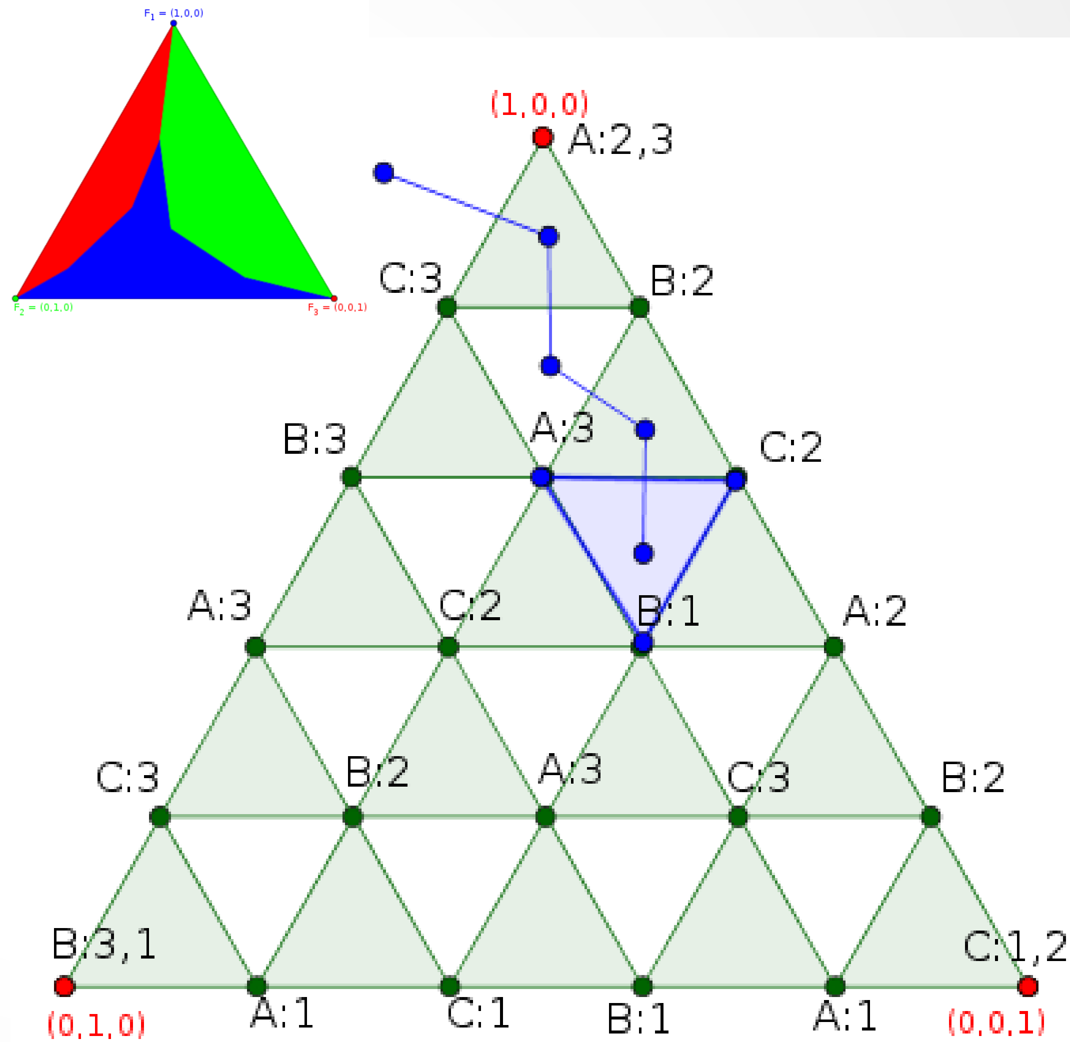

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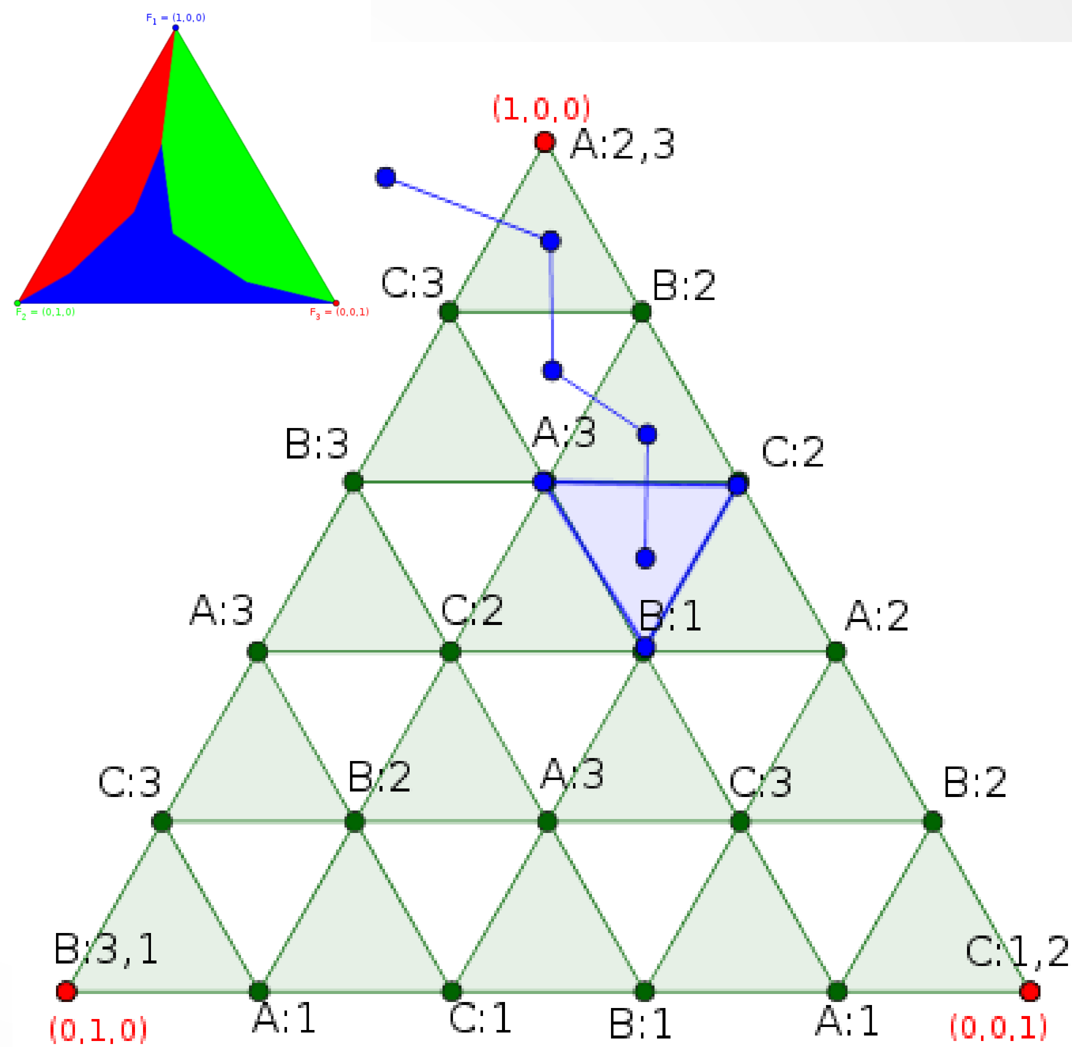

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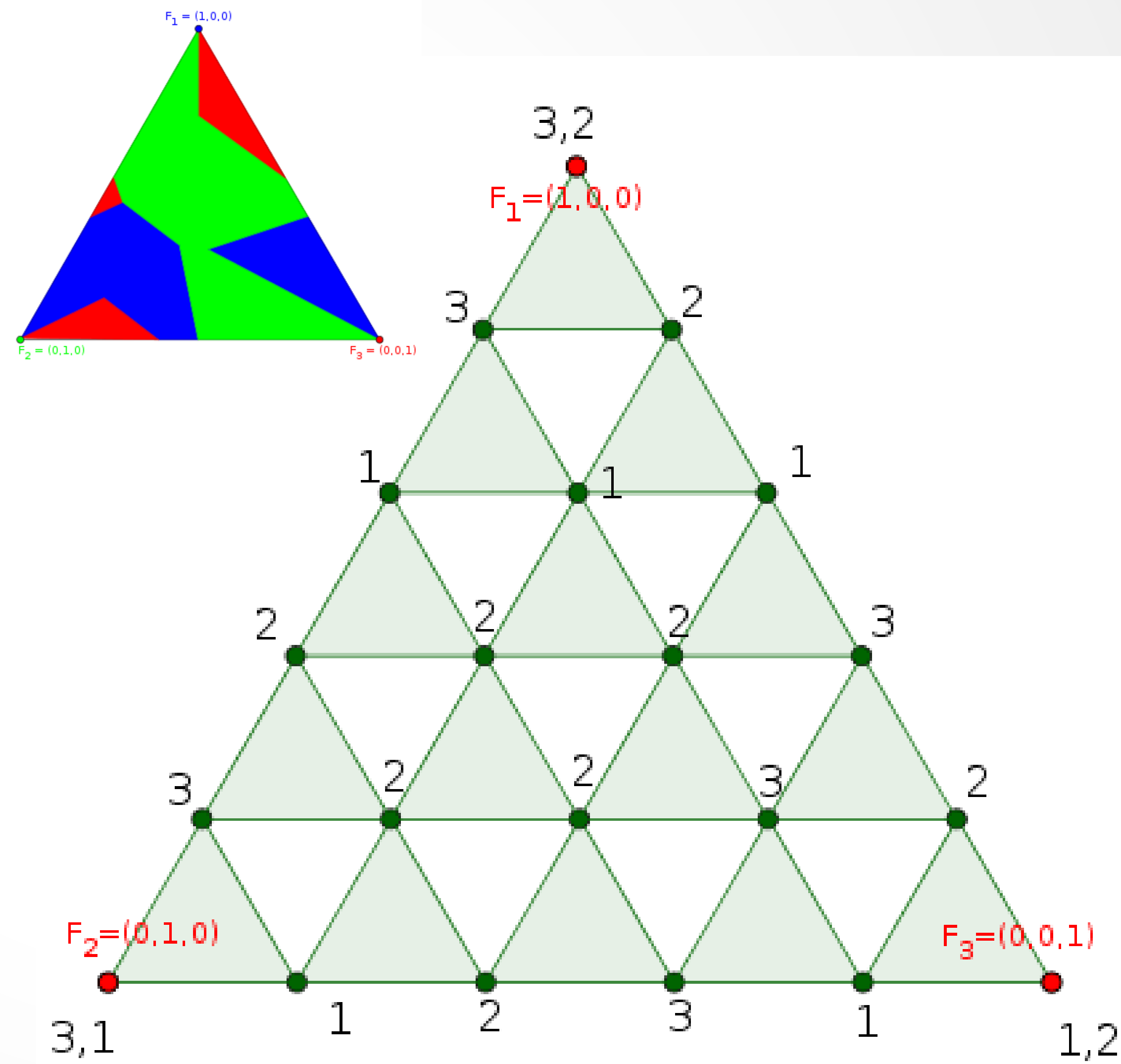
Corollary (Su 1999): when valuations are also continuous, an envy-free division exists.

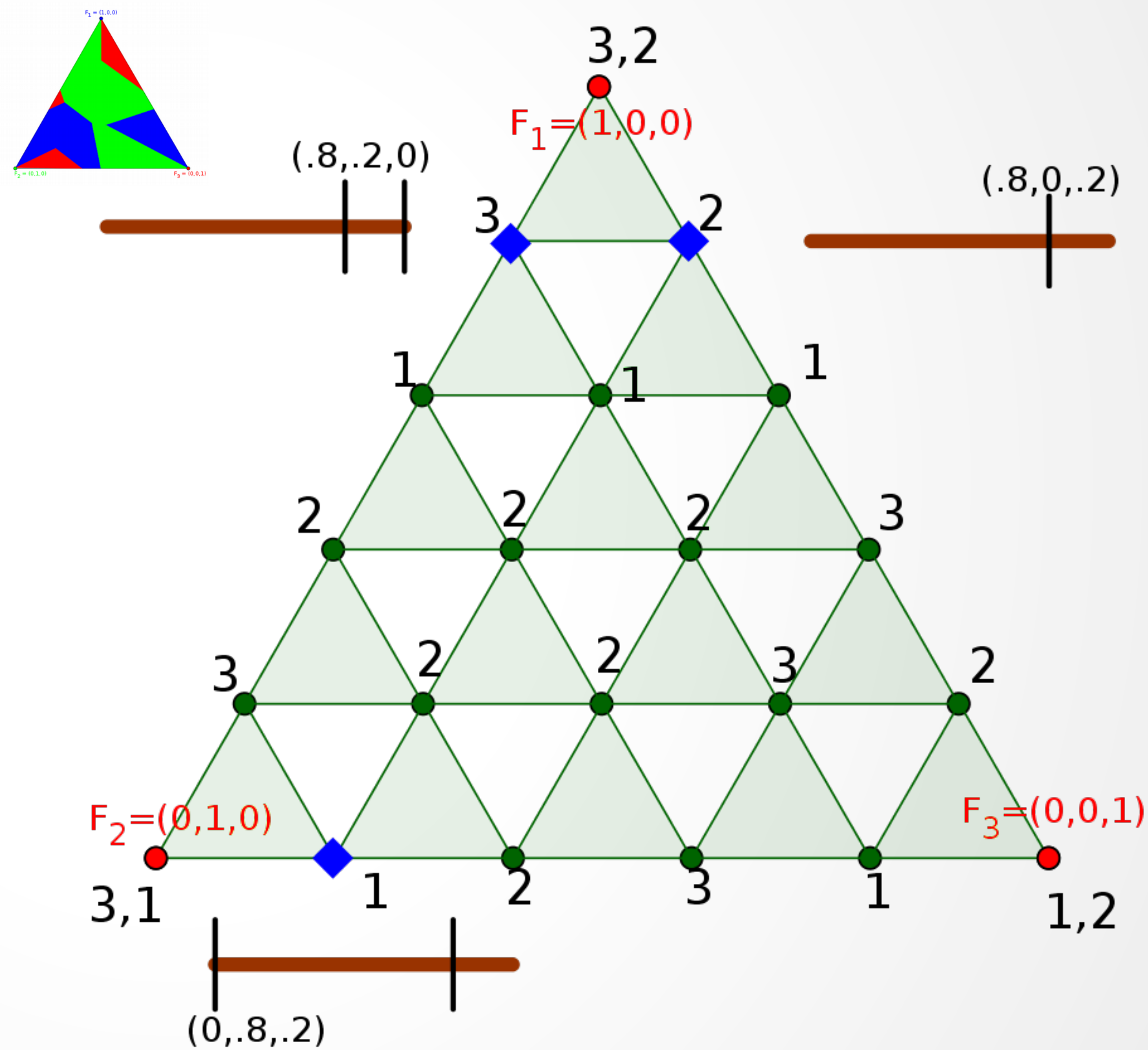
Triangulation – General Agents

In general, the conditions for Sperner's lemma are **not** satisfied.

What can we do?

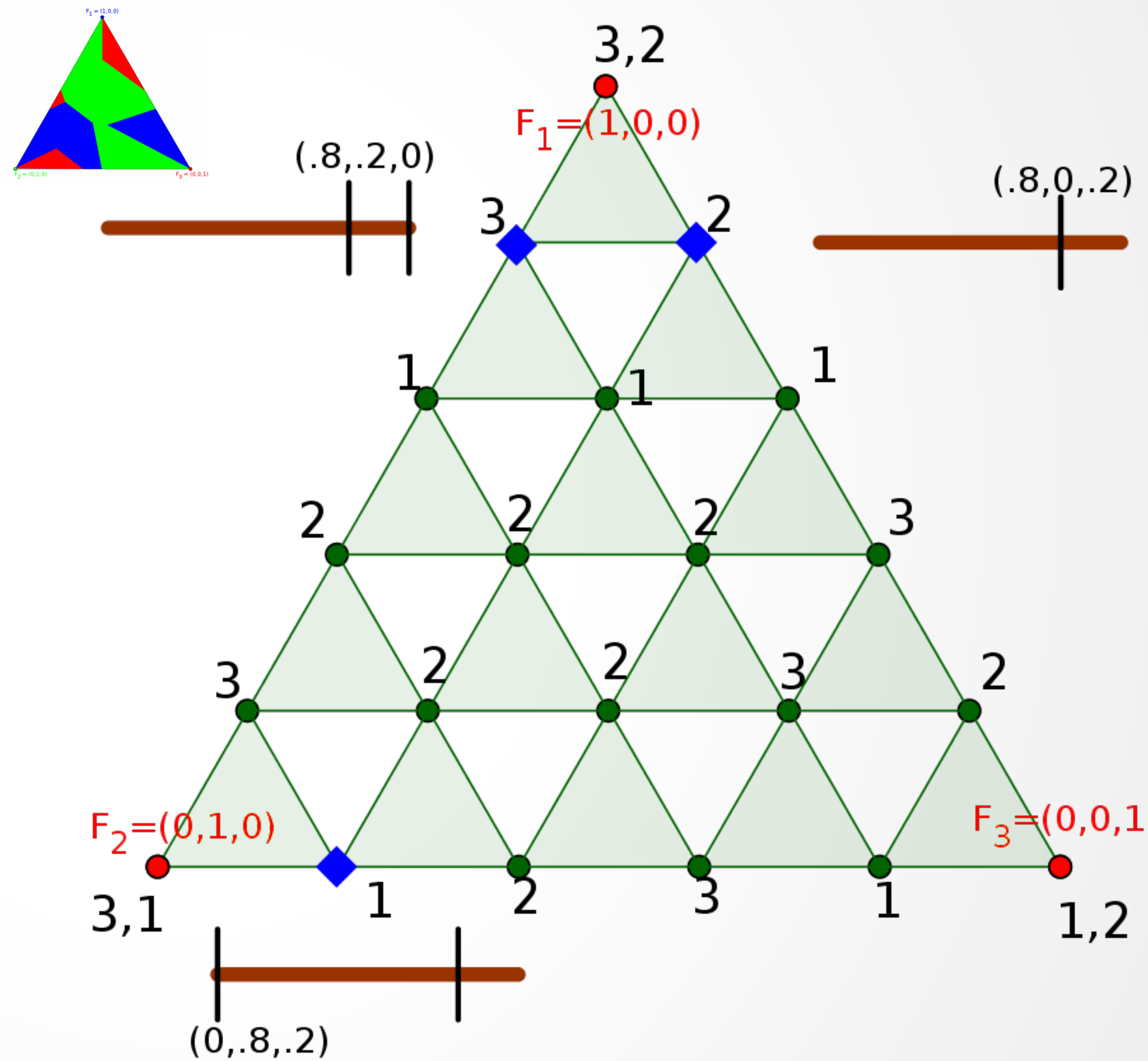


Boundary Permutation Condition



Boundary Permutation Condition

Definition: Two vertices in the simplex are called *friends* if they have the same ordered list of non-zero coordinates.



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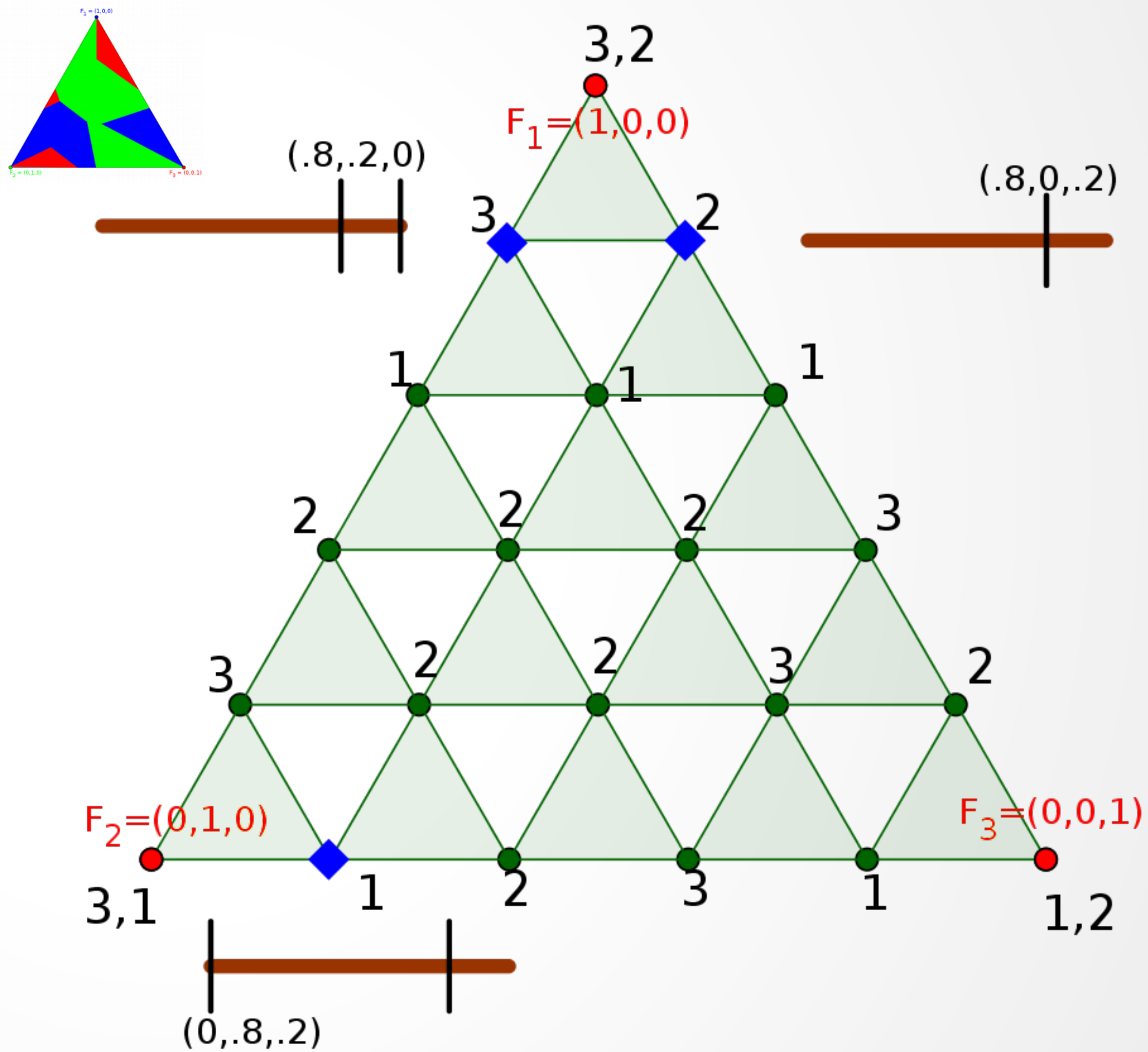

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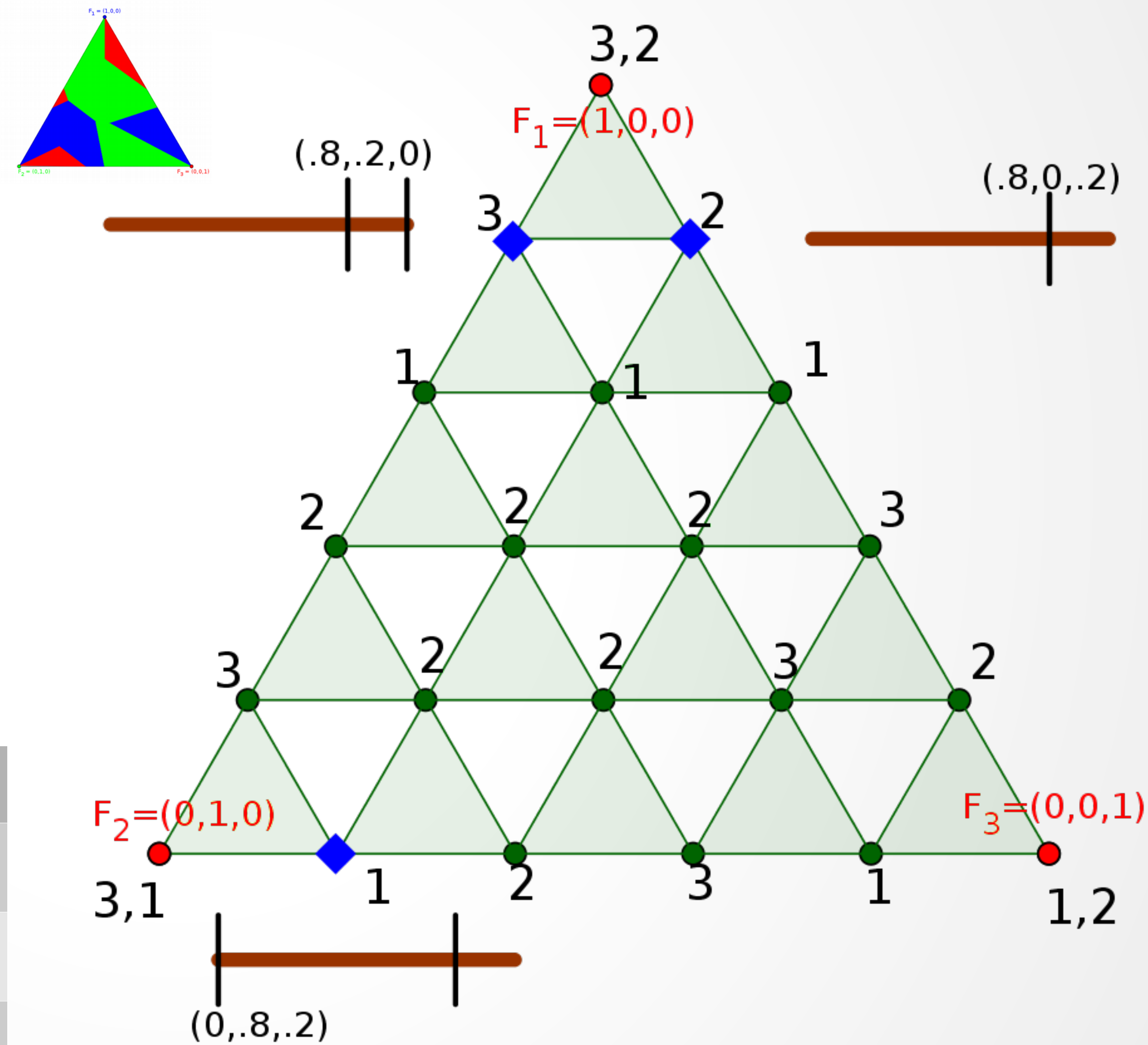
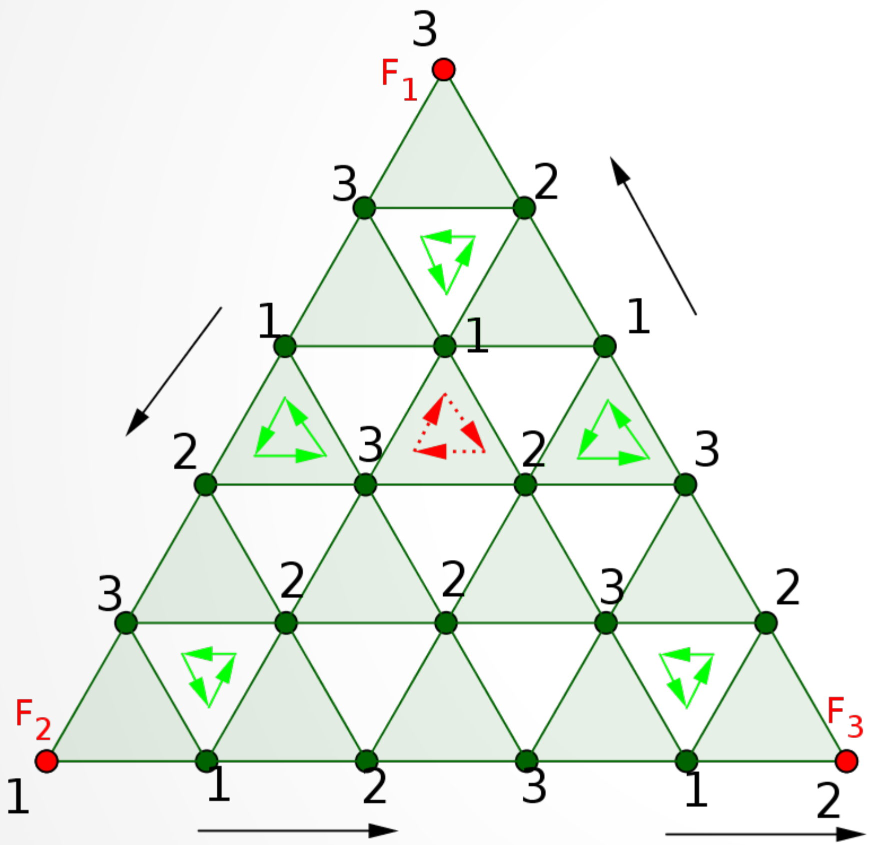
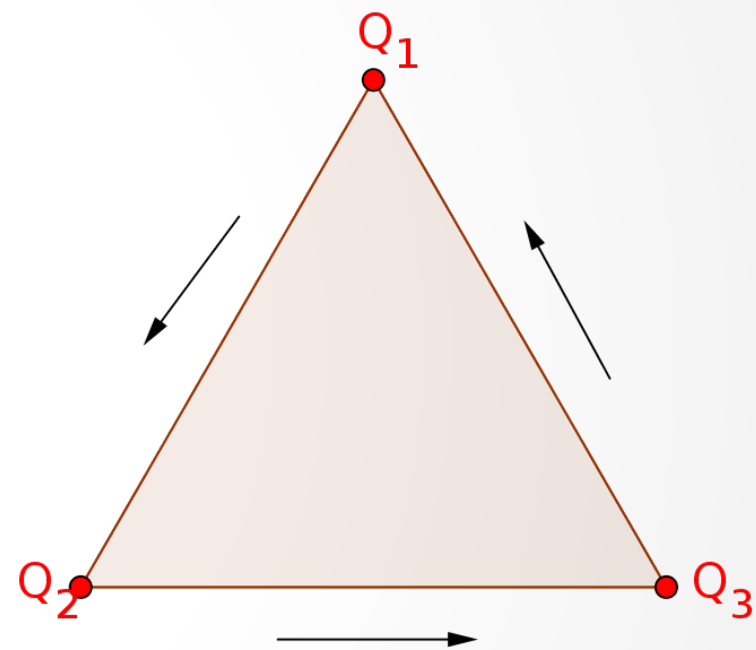


Degree of Labeling

Labeling ≡ mapping from triangulation vertices to vertices of Q

(follows Musin 2014)





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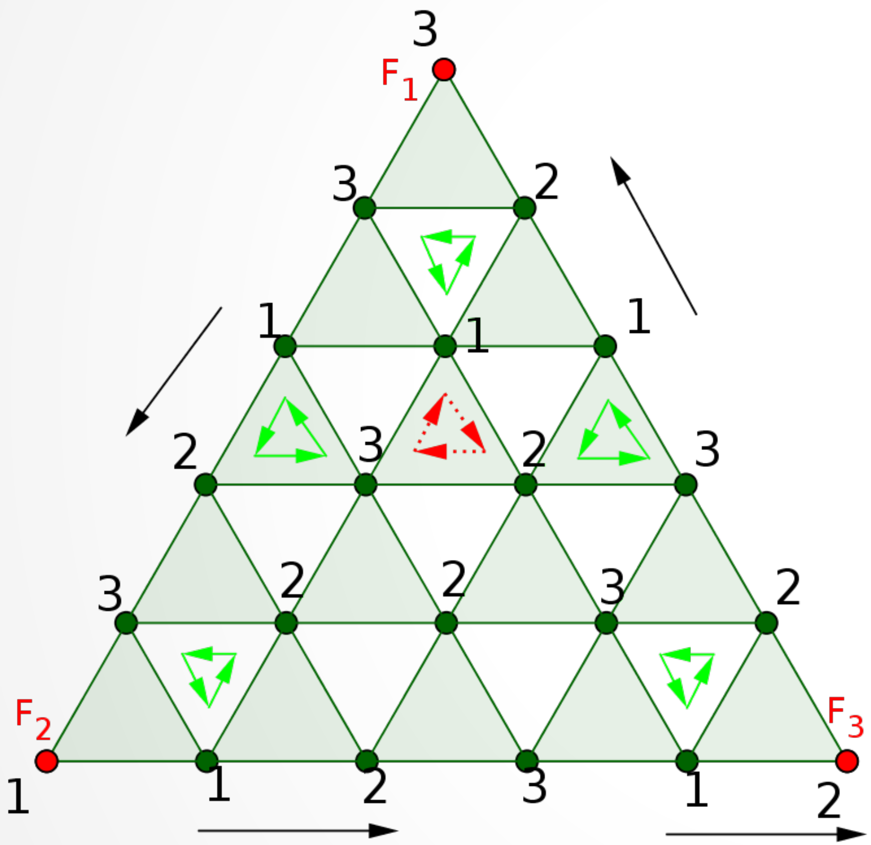

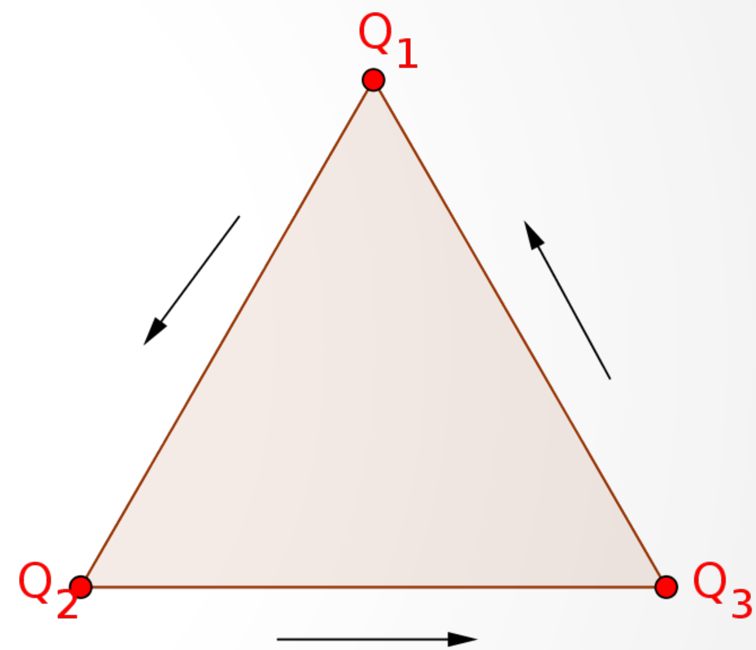

Degree of mapping = net number of rounds *(CCW=positive)*.

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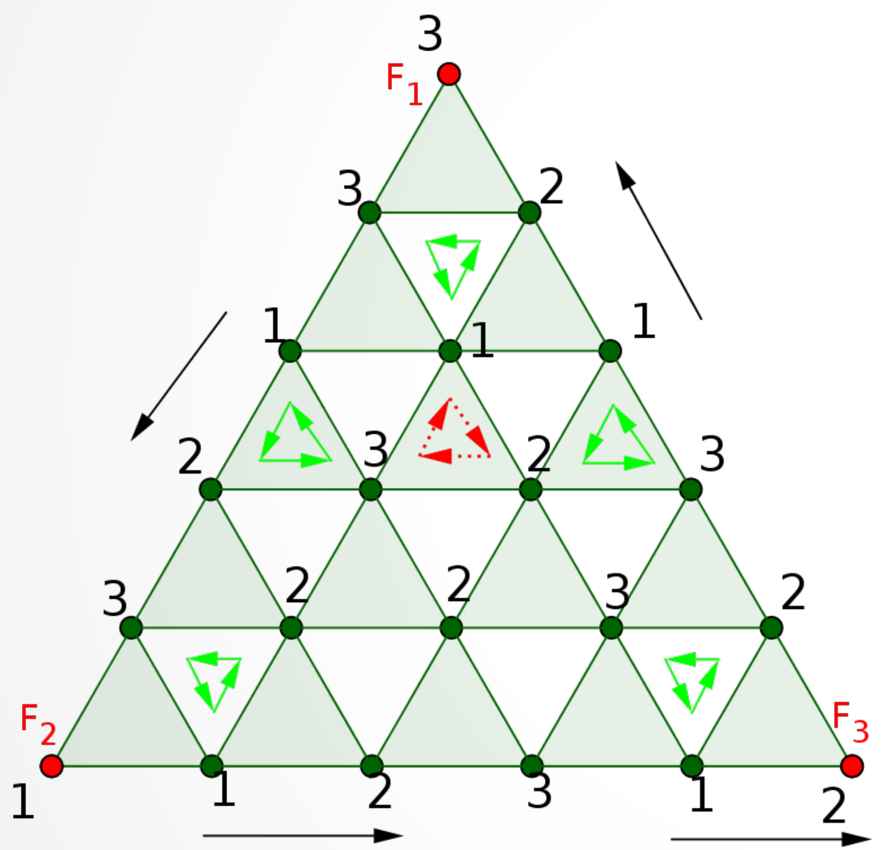
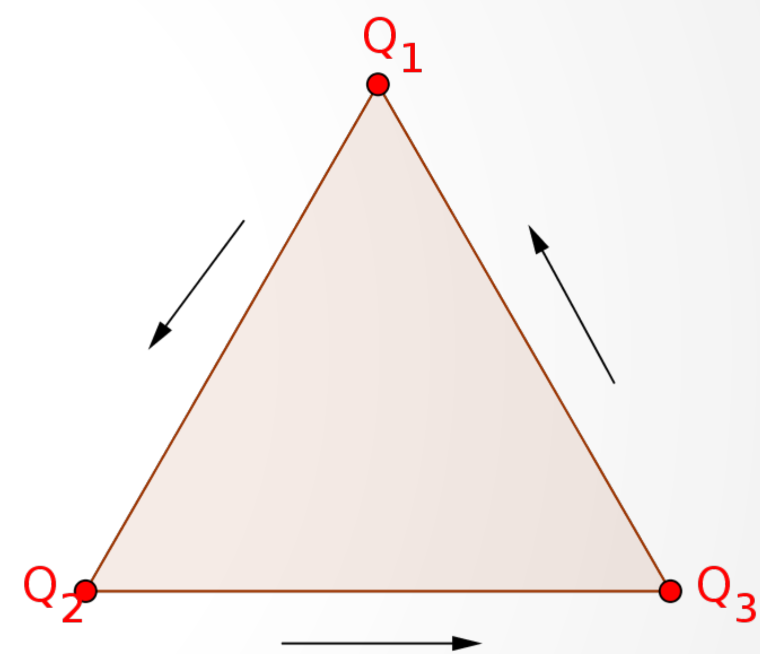




Degree of mapping = net number of rounds *(CCW=positive)*.

Lemma: degree on boundary = degree in interior.

Steps in Existence Proof

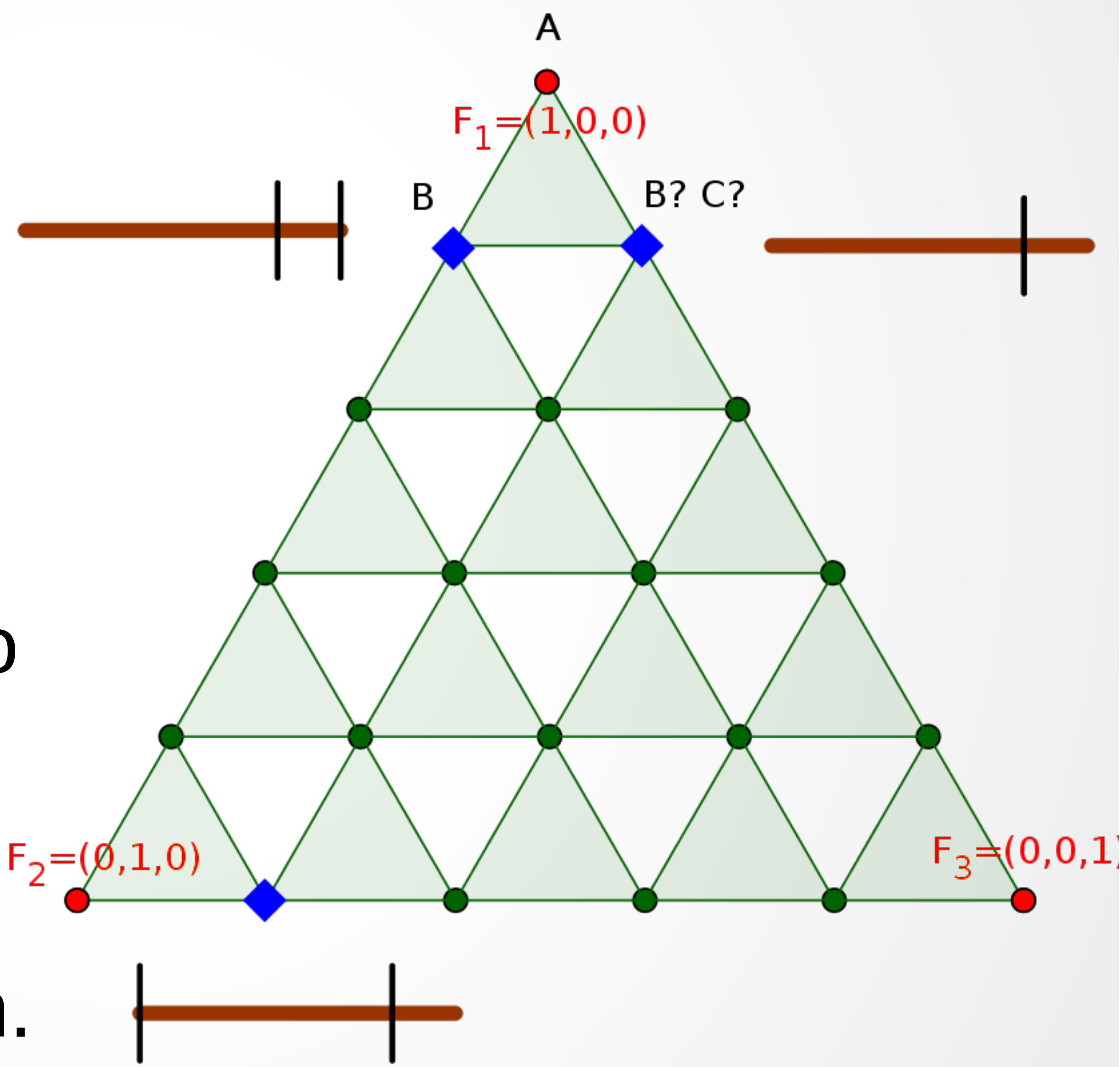
Step	Proved for
1. n agent-labelings with permutation condition → Combined labeling with permutation condition	Any n
2. Permutation condition → Nonzero boundary degree	$n = 3$
3. Boundary degree = Interior degree	Any n (?)

Step 1: n labelings → 1 labeling

We need to assign owners to vertices s.t.:

- In each sub-simplex, each vertex belongs to a **different** owner.
- Friends are assigned to **the same** owner.

Does not work with the equilateral triangulation.



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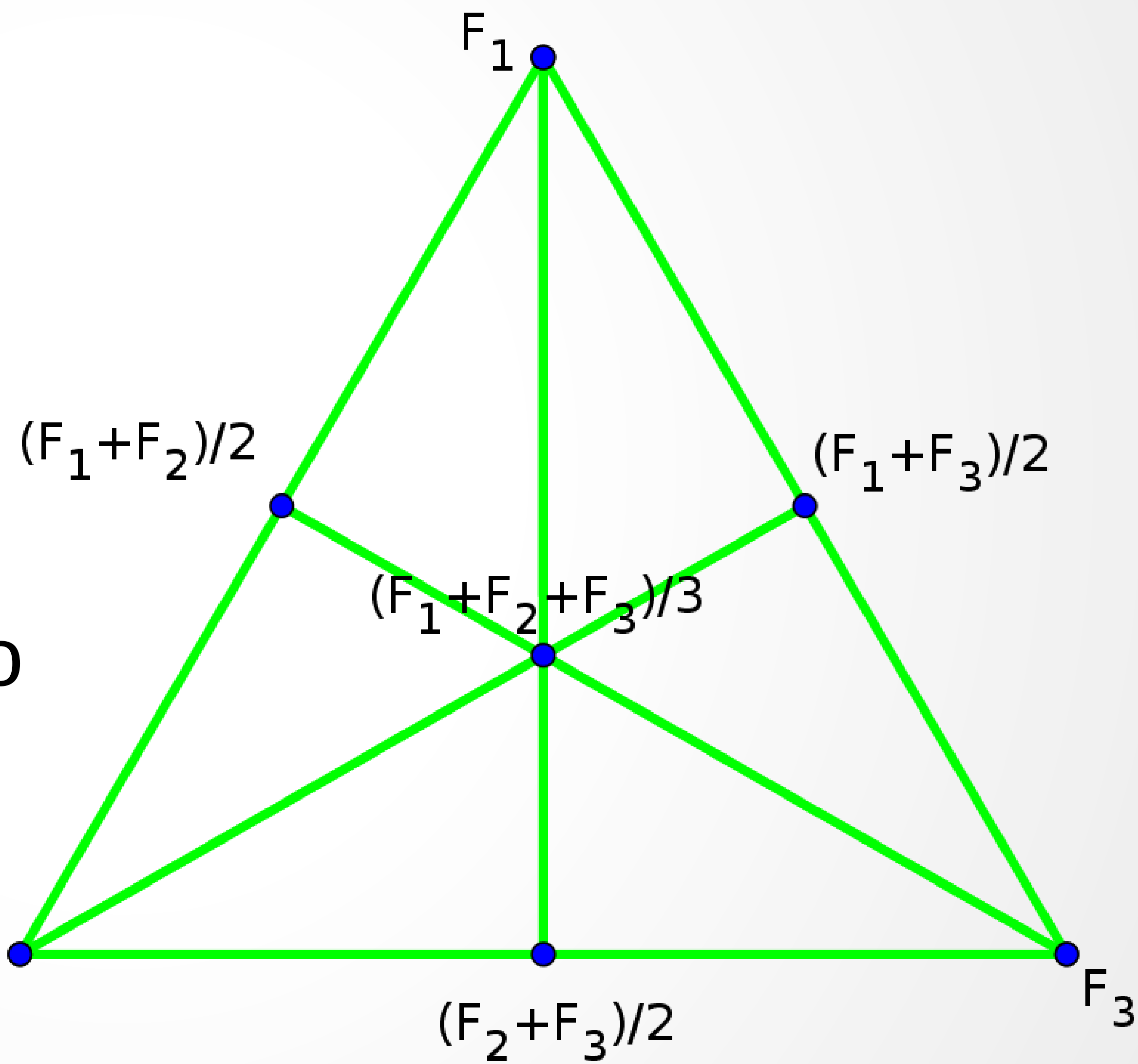

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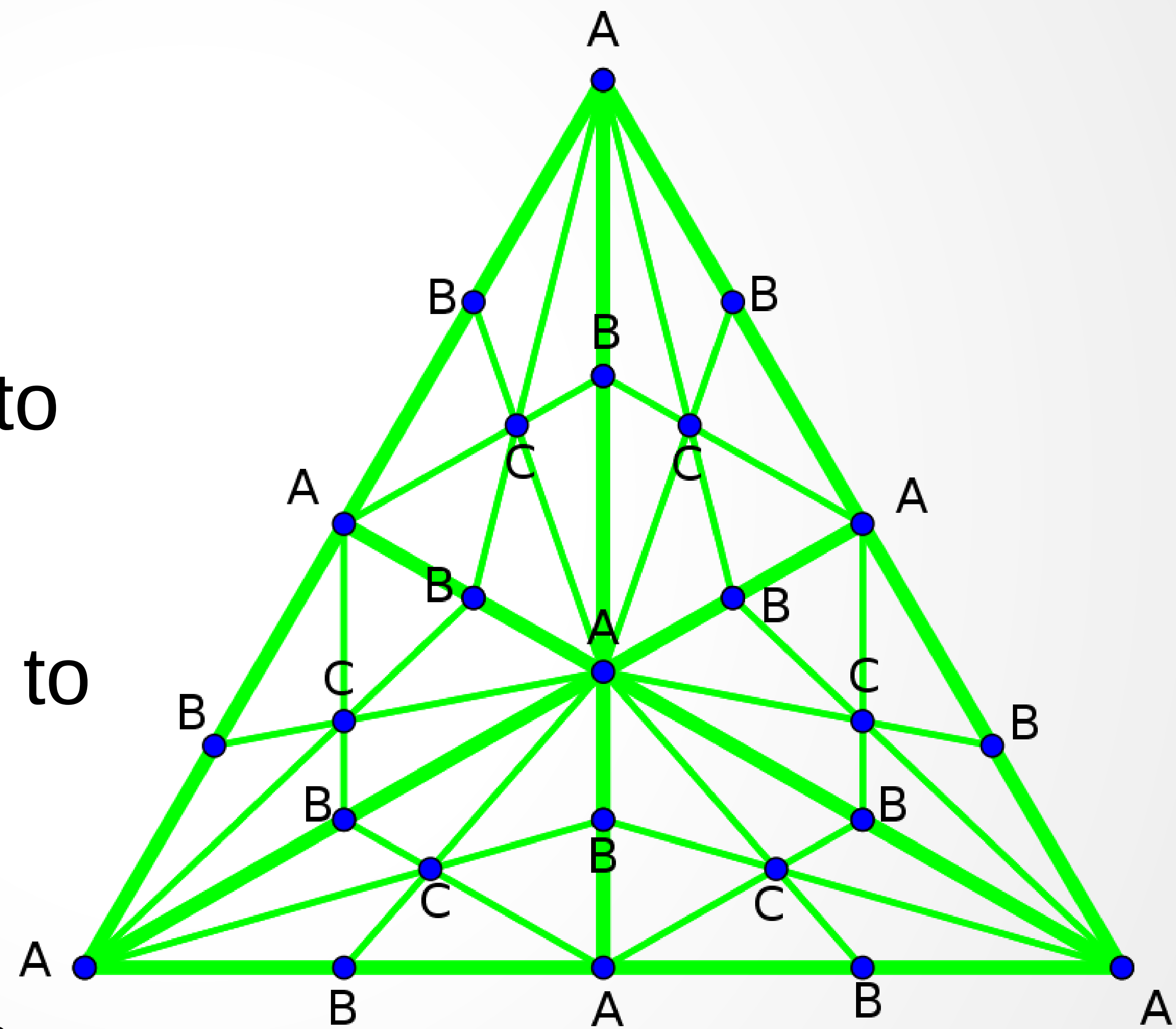

Lemma: it works with *barycentric triangulation*

Step 2: Permutation → Boundary degree

Permutation condition:

Pref:	Left	Right	Empty	
F_{12}	1	2	3	*Even*
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Agent condition: Either:
(+) In each main-vertex i, the label is i, or:
(-) In each main-vertex i, the label can be anything but i.

Lemma: *When n=3*, if labeling satisfies
permutation condition and *agent condition*,
then labels on main vertices can be chosen such that:
boundary-degree mod 3 <> 0.

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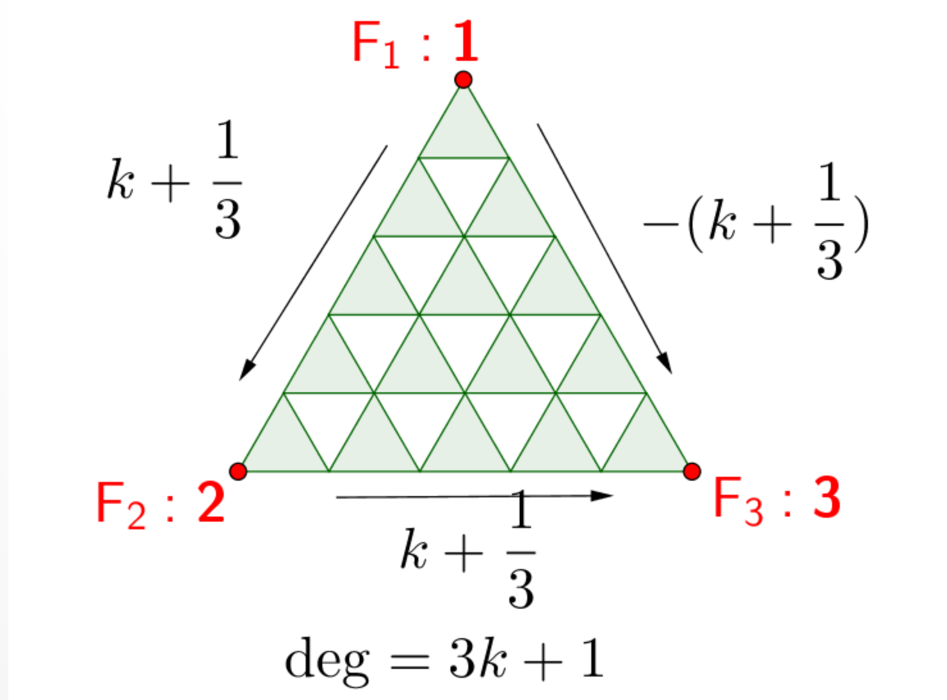
Proof:



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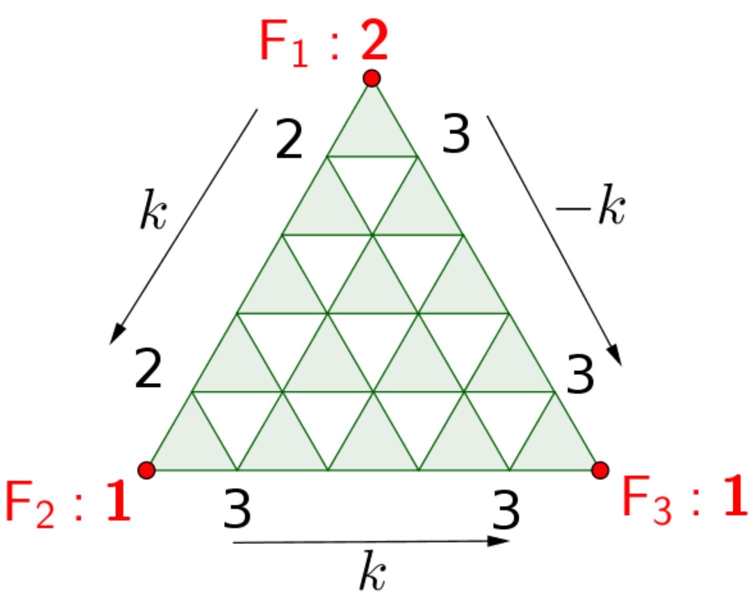
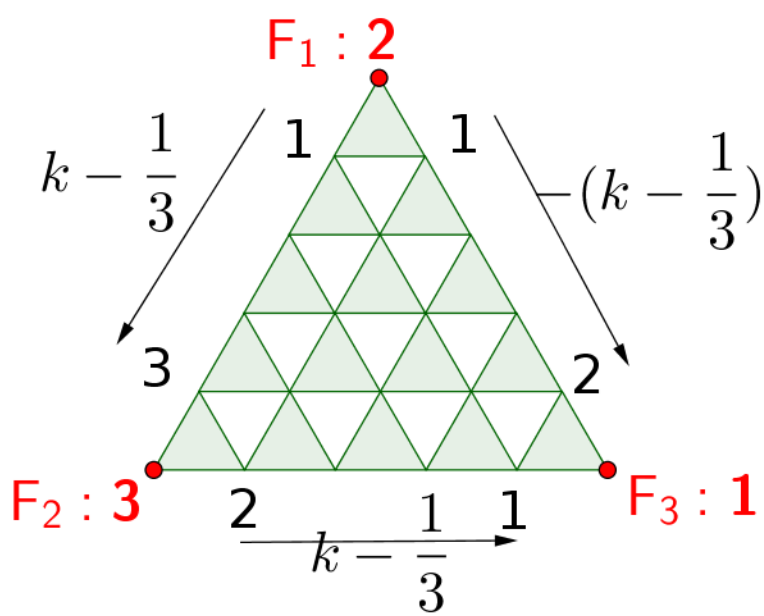
2 of 9 cases shown below:



$$\deg = (3k) - 2/3 - 1/3 = 3k - 1$$



$$\deg = (3k - 1) - 1/3 + 1/3 = 3k - 1$$

Step 3: Boundary degree = Interior degree

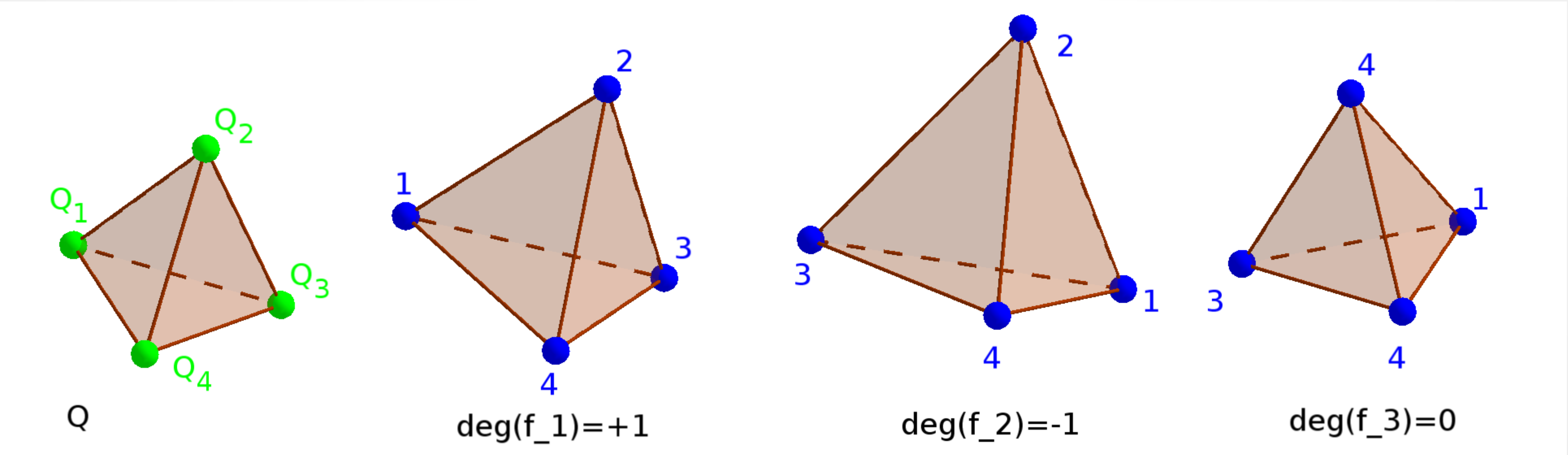
Definition:

Degree of labeling of an n-simplex in $R^{n\text{-}1}$

= sign of determinant of affine transformation to Q

= +1 if onto&no reflection, -1 if onto&one reflection,

0 if not onto.



Step 3: Boundary degree = Interior degree

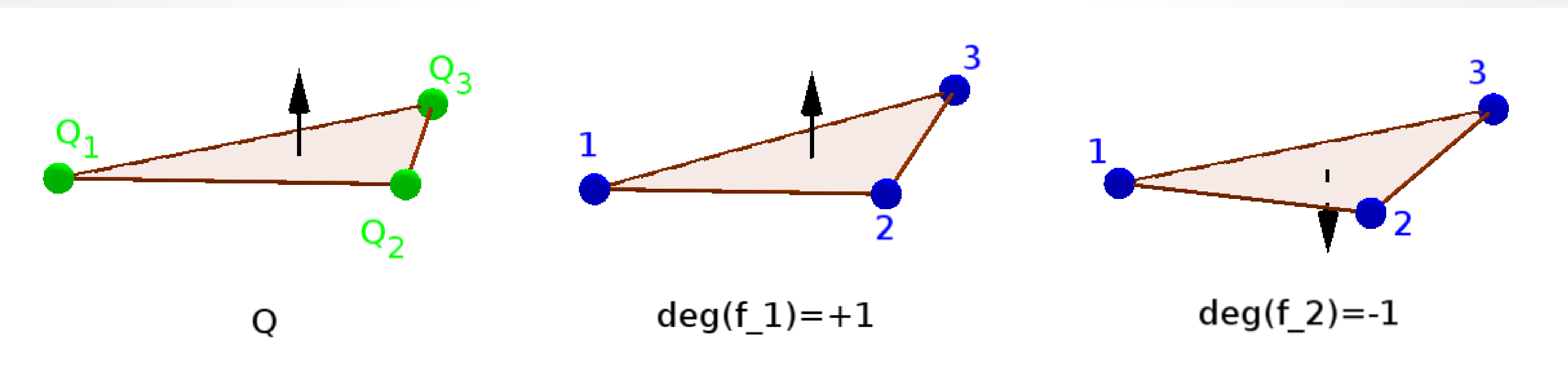
Definition:

Orientation of an $(n\text{-}1)$-simplex in $R^{n\text{-}1}$

= one of its two adjacent half-spaces.

Degree of labeling of an $(n\text{-}1)$-simplex in $R^{n\text{-}1}$

= sign of determinant of *any* affine transformation to Q that perserves the orientation.



Step 3: Boundary degree = Interior degree

Lemma:

Degree of a labeling of an n-simplex in $R^{n\text{-}1}$,

= sum of degrees on each face oriented *inwards:*

Step 3: Boundary degree = Interior degree

Lemma:

Degree of a labeling of an n-simplex in R^{n-1},

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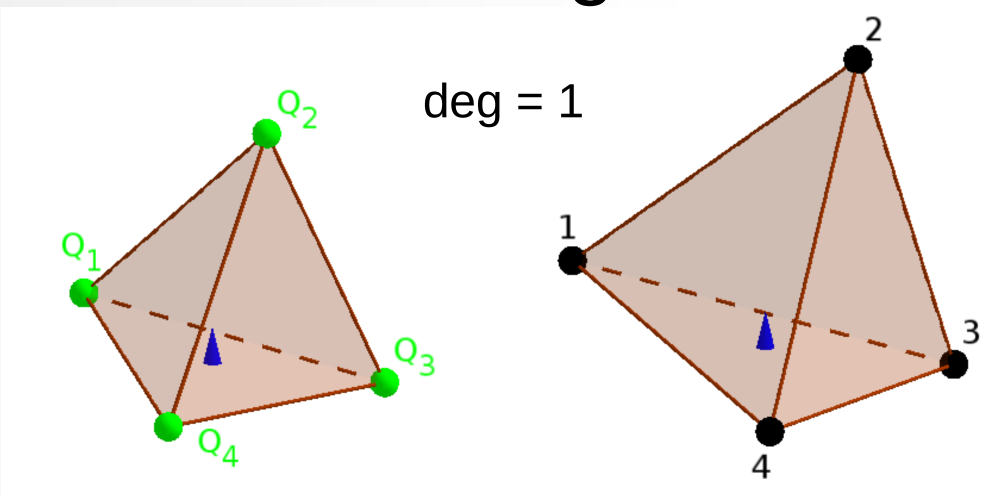

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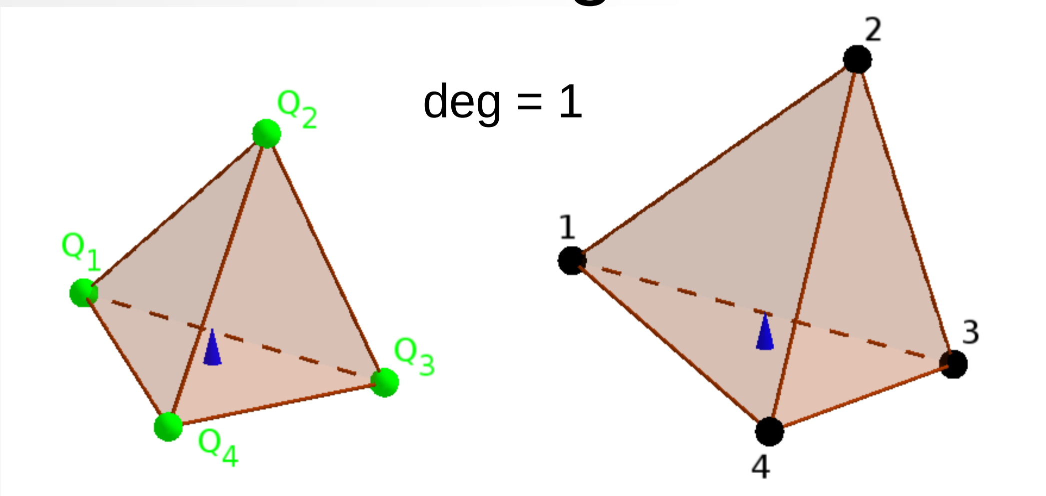

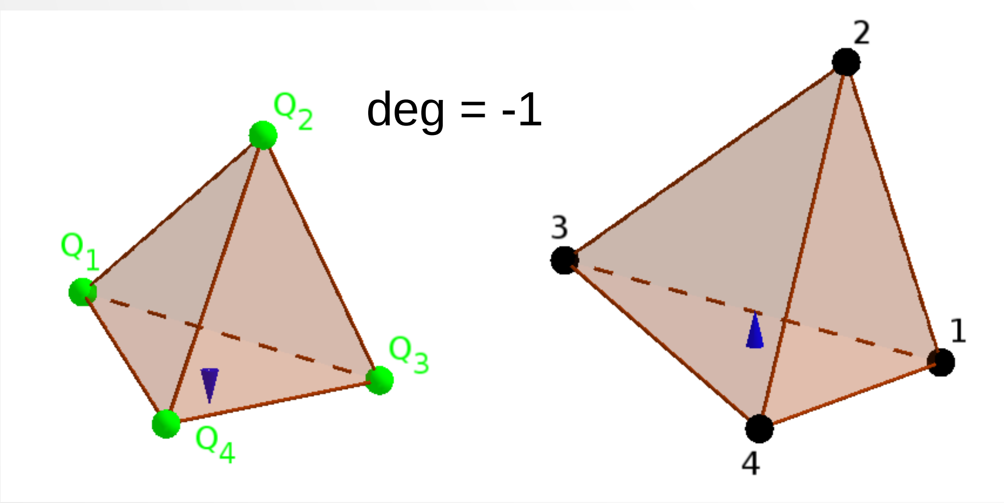

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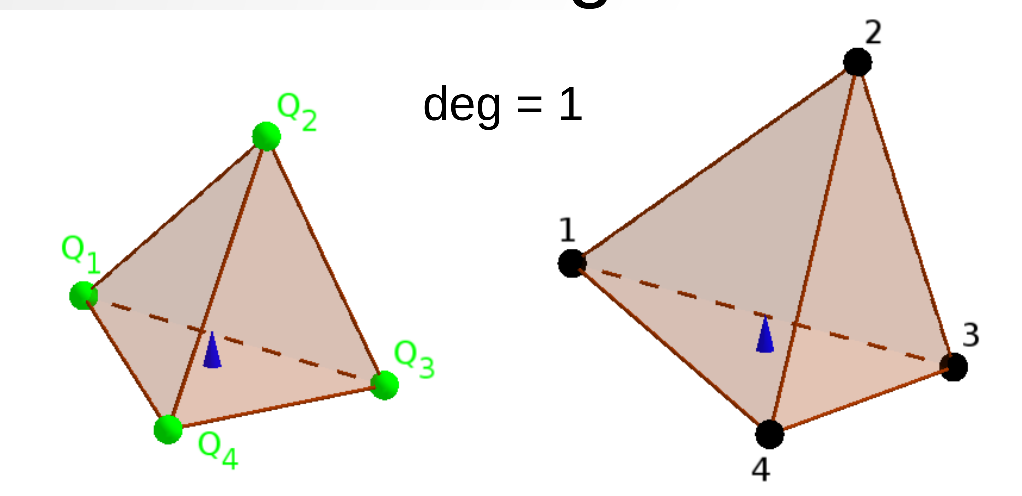

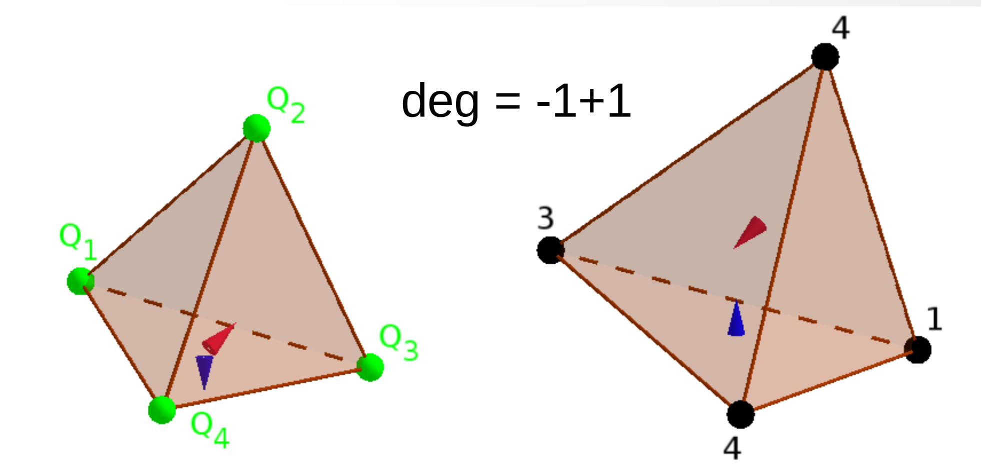

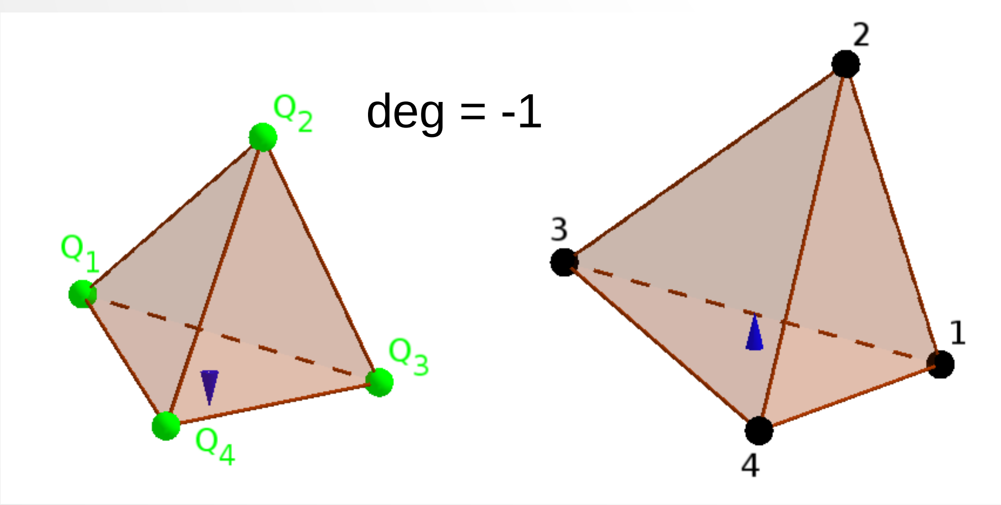

Step 3: Boundary degree = Interior degree

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Degree of a labeling of an n-simplex in R^{n-1},
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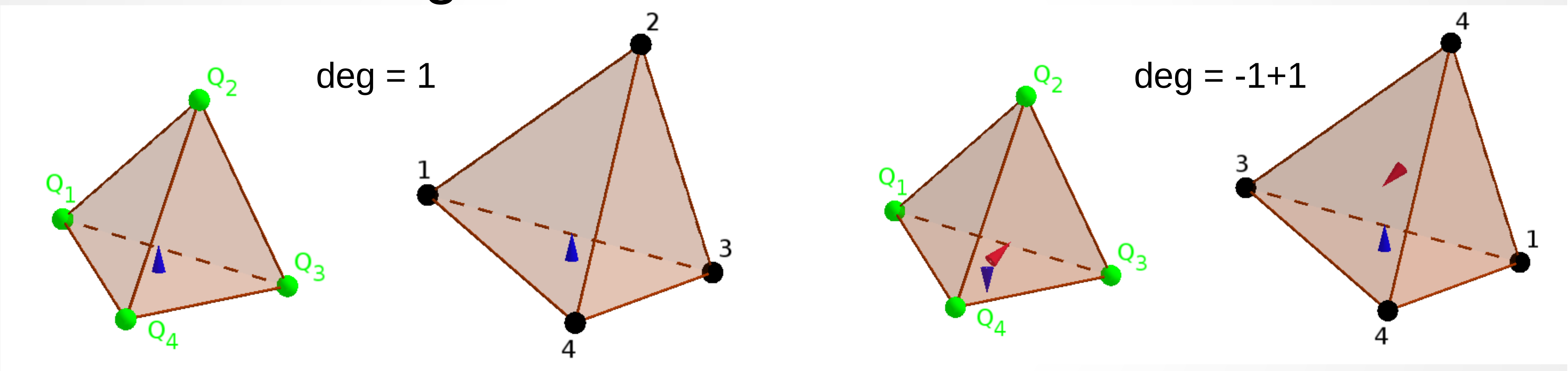
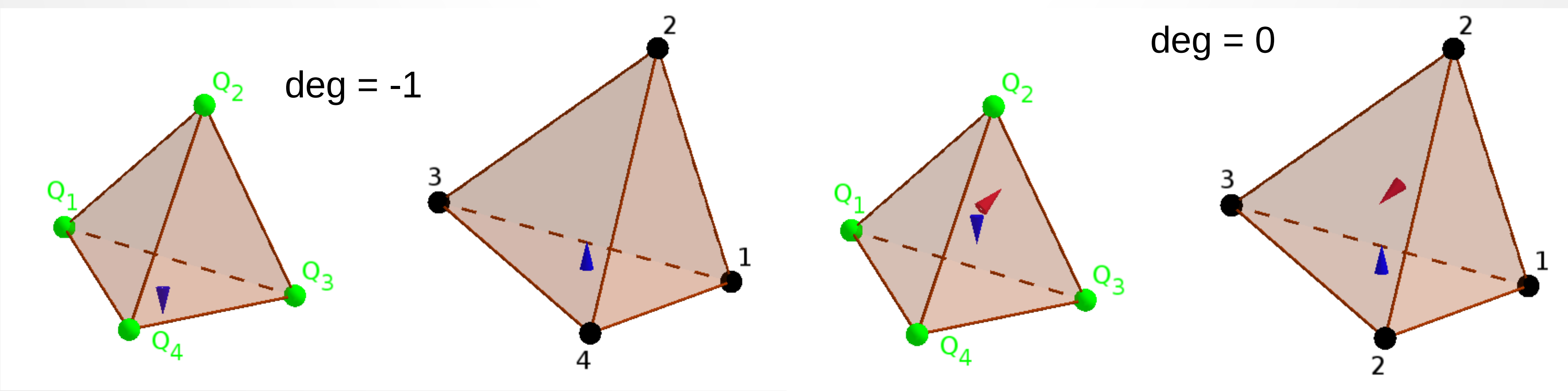
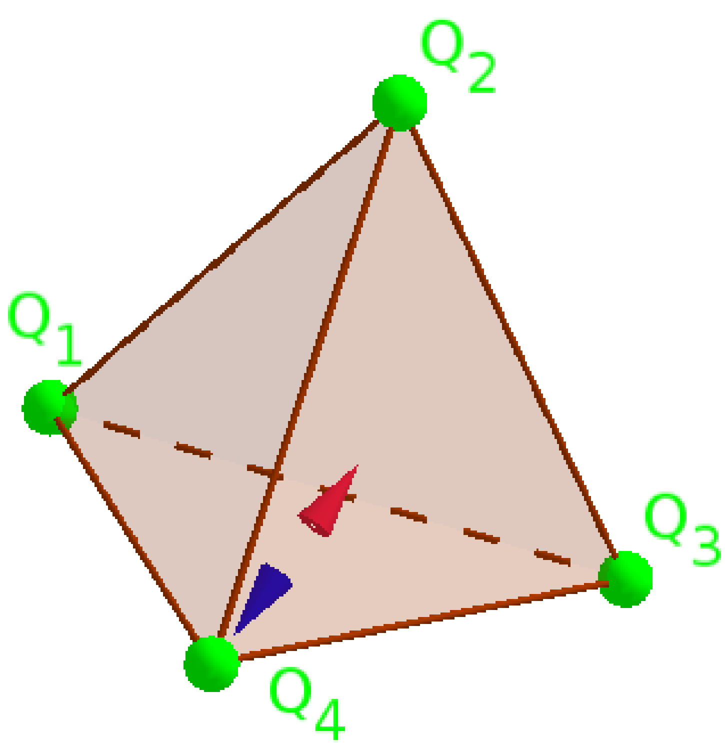


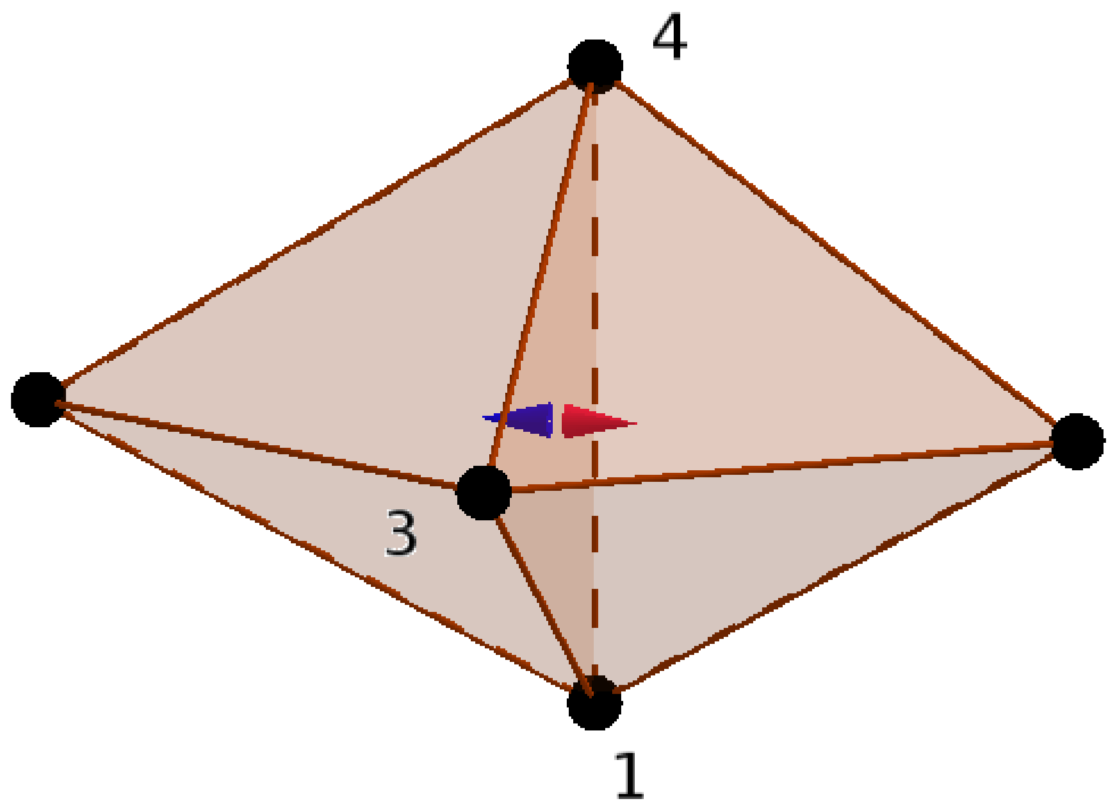


Step 3: Boundary degree = Interior degree

Lemma:

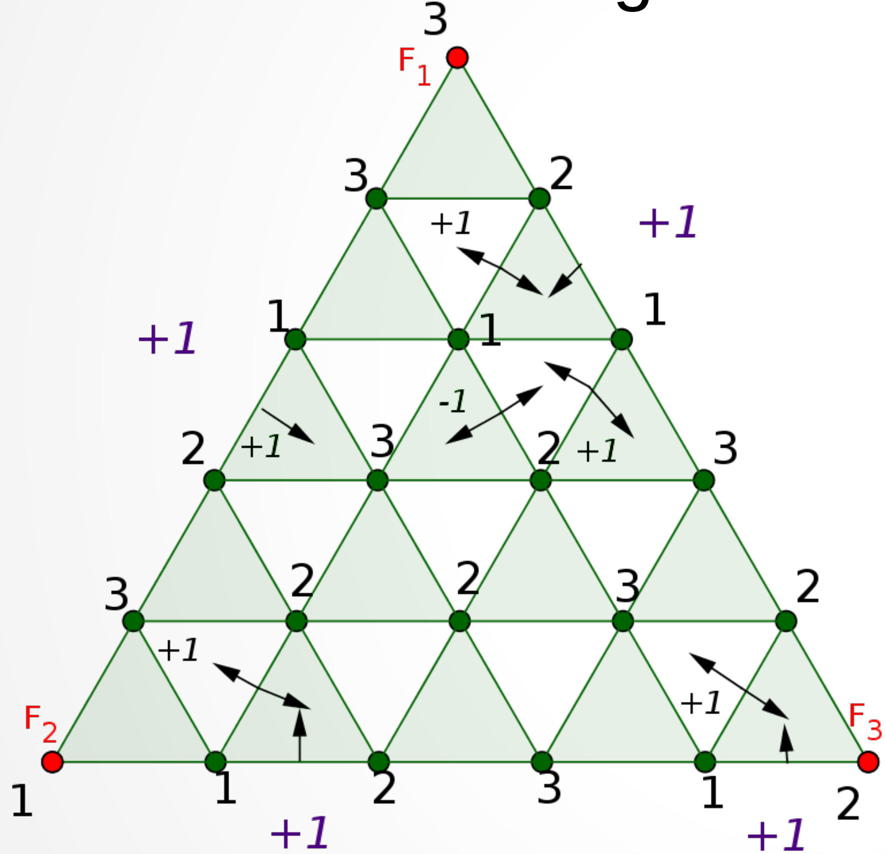
Sum of degrees of simplices in triangulation
= sum of degrees on each *boundary* face,
– since the internal faces cancel out:

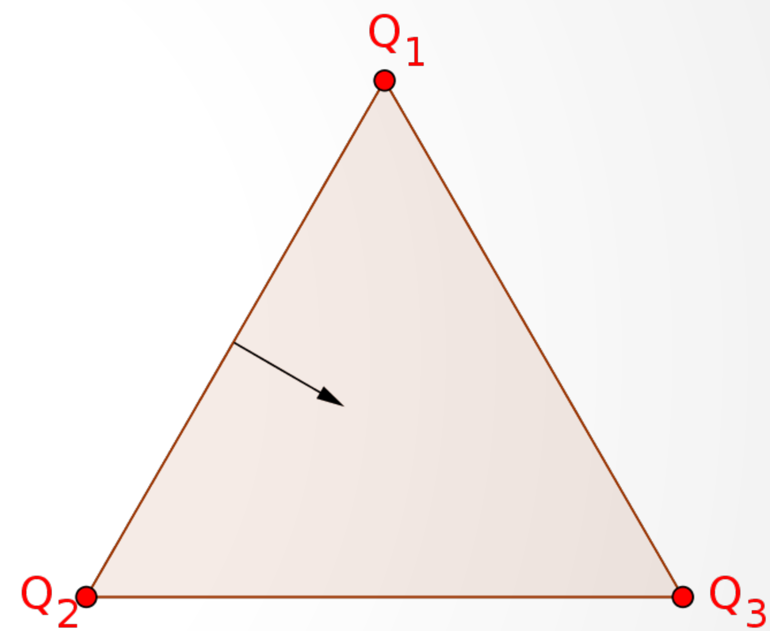




Step 3: Boundary degree = Interior degree

Definition: degree of triangulation labeling
= sum of degrees of each sub-simplex labeling.





Step 3: Boundary degree = Interior degree

Definition: degree of triangulation labeling
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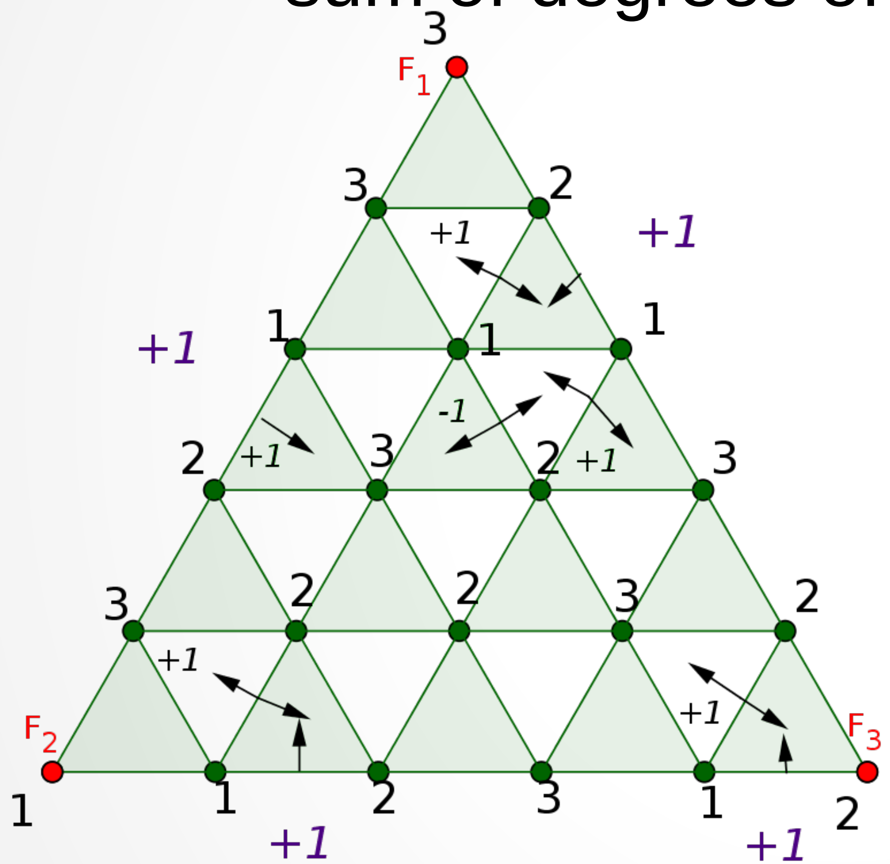
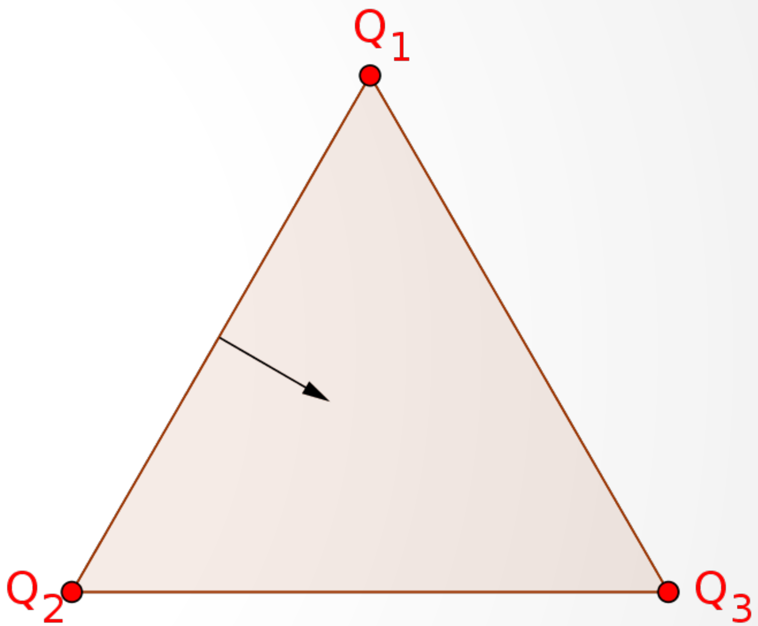




Lemma: interior degree = sum of degrees on faces
= sum of degrees on faces of boundary = boundary degree.

Conclusion

Step	Proved for
1. n agent-labelings with perm. condition → Combined labeling with perm. condition	Any n
2. Permutation condition → Nonzero boundary degree	$n = 3$
3. Boundary degree = Interior degree	Any n (?)

Theorem: for 3 agents with continuous valuations,
an envy-free connected division exists
for arbitrary mixed valuations.

Open question

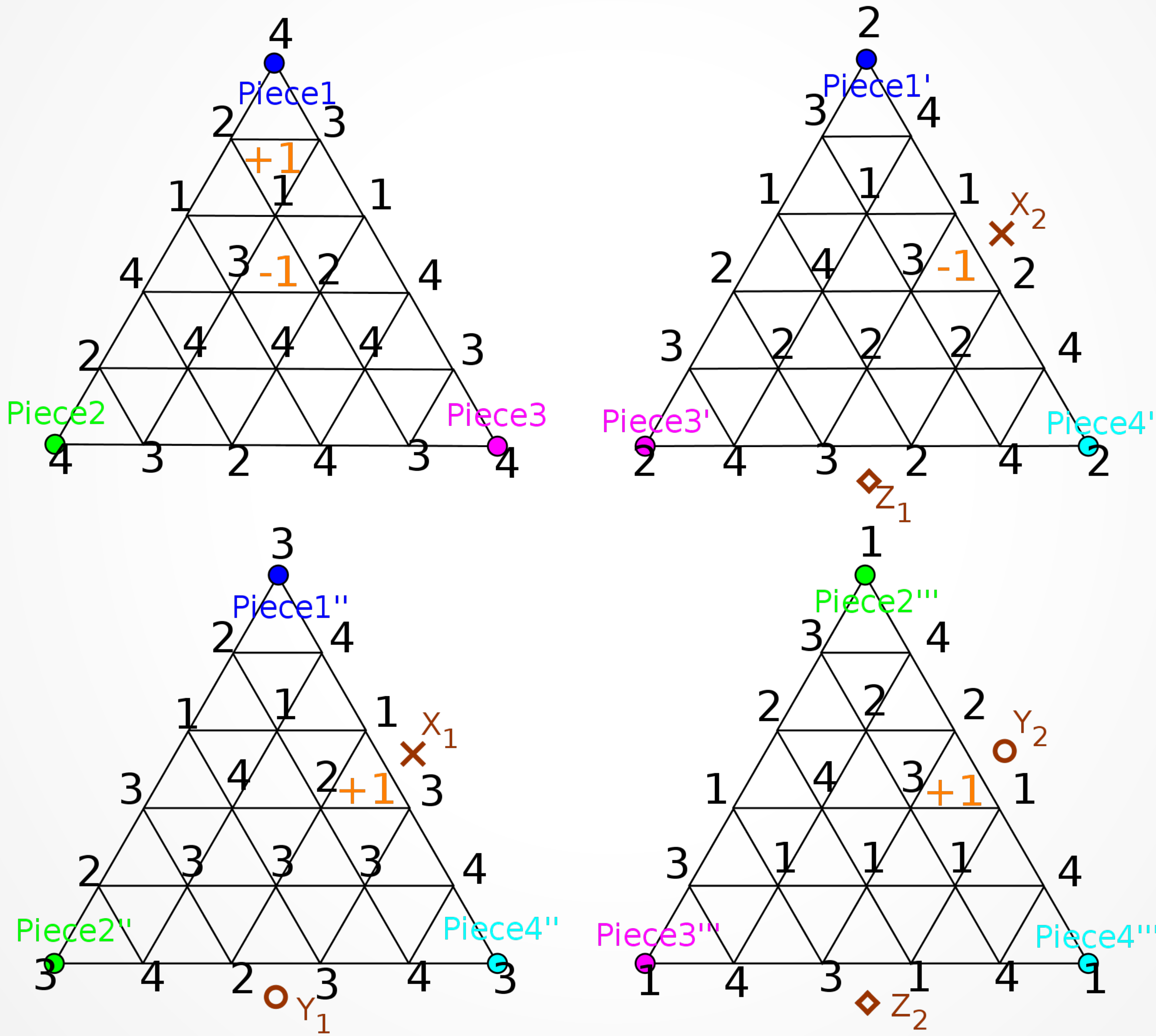
Permutation condition for 4 or more agents:

Pref:	Left	Middle	Right	Empty	
F_{123}	1	2	3	4	*Even*
F_{124}	1	2	4	3	*Odd*
F_{134}	1	3	4	2	*Even*
F_{234}	2	3	4	1	*Odd*

Conjecture: If labeling satisfies
permutation condition and *agent condition*,
then boundary-degree mod n <> 0.

If conjecture is true, then connected envy-free division exists for arbitrary mixed valuations!

Open question



Dividing Goods that are Bads

(Midrash Rabba, Genesis 33:1)

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