"DIVIDE THE LAND EQUALLY" (Ezekiel 47:14)

Fair Division among Families

Erel Segal-Halevi

ARIEL UNIVERSITY

Based on joint works with:

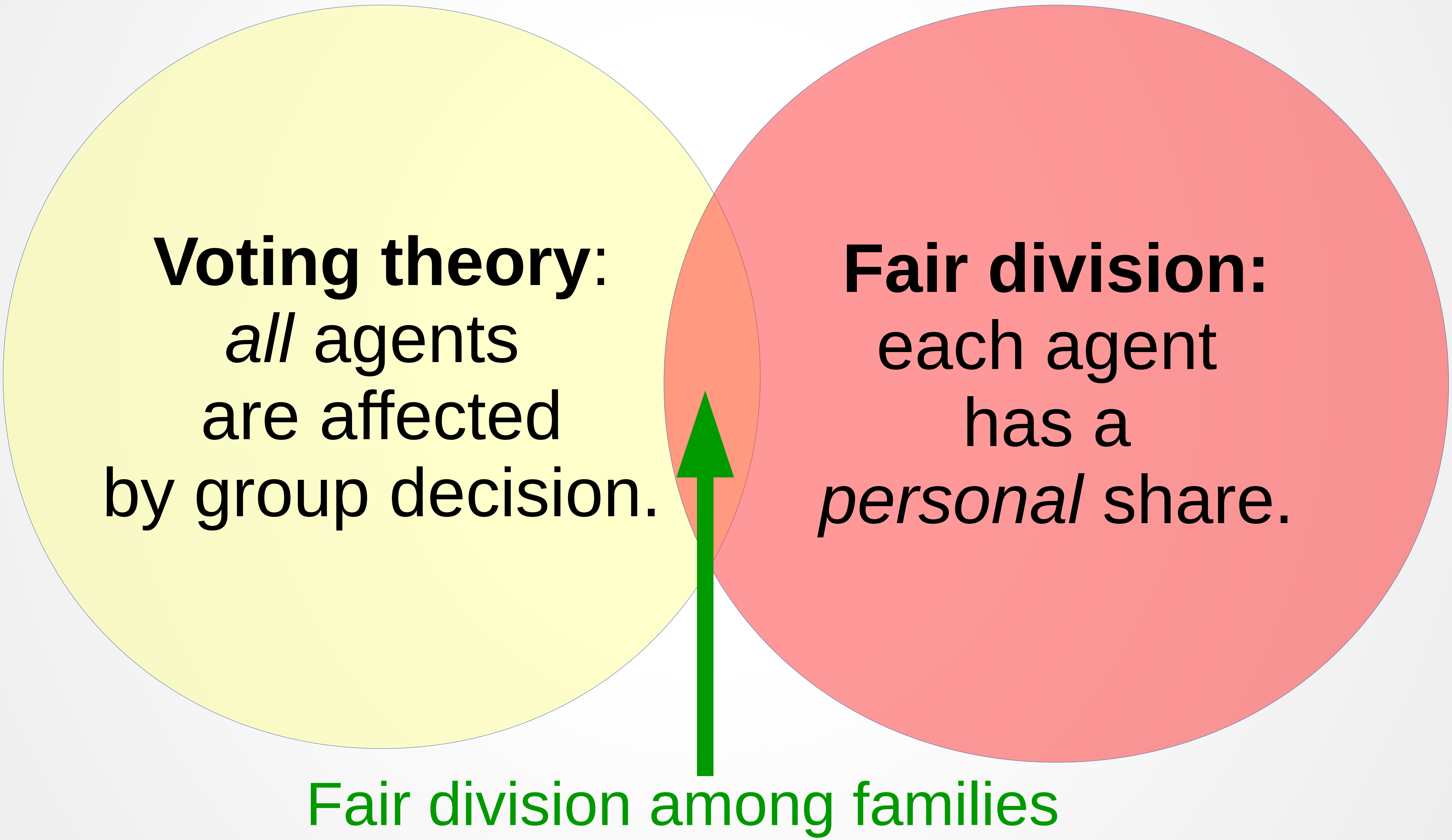
- Shmuel Nitzan – Bar Ilan University
- Warut Suksompong – Oxford University
- Sophie Bade – Royal Holloway University of London

Individual vs. Family Goods

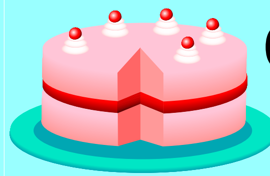
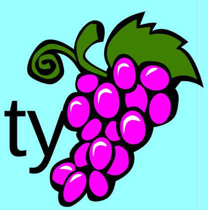
Different preferences;
Different shares;
Each agent should believe his share is “good enough“.

Different preferences;
Same share;
Each agent should believe his **family‘s** share is “good enough“.

Social Choice Theory



Fair Division Settings

Resource type	Example	Challenge
1. Heterogeneous, divisible resource	 cake, land	Fair and **connected**.
2. Homogeneous, divisible resources	fruits, electricity 	Fair and **Pareto-optimal**.
3. Indivisible goods	 jewels, houses	**“Almost“** fair.

1. Heter. div. – Individuals 

Fairness in an economy of individuals:

- *Envy-free (EF)*: each individal's utility in his share ≥ his utility in any other share.
- *Proportional (PR)*: each individal's utility in his share is ≥ (cake utility) / (# individuals).

Theorem (Stromquist, 1980):

- For any number of individuals, there exists a connected + EF + PR allocation.

1. Heter. div. – families

Fairness in an economy of families:

- *Envy-free (EF)*: each individual's utility in his family's share ≥ utility in another family's share.
- *Proportional (PR)*: each individual's utility in his family's share ≥ (cake utility) / (# families).

Theorem (with Shmuel Nitzan):

- There might be no allocation that is both connected and EF and/or PR.

1. Heter. div. – families

Theorem 1-: There are instances with 2 families where no connected allocation is EF/PR.

Proof: There are a couple and a single. Each individual wants a distinct segment of the cake:

In any connected division, at least one individual gets a utility of 0.

2. Homog. div. – individuals

Fairness in an economy of individuals:

- *Envy-free (EF)*: each individual prefers his share to the shares of all other agents.
- *Fair-share guarantee (FS)*: each individual prefers his share to an equal split of resources.

Theorem (Varian, 1974):

- If all preferences are monotone and convex, then PO+EF+FS are compatible.

2. Homog. div. – families

Fairness in an economy of families:

- *Envy-free (EF)*: each individual prefers his *family*'s share to shares of all other families.
- *Fair-share guarantee (FS)*: each individual prefers his *family*'s share to the equal split.

Theorem (with Sophie Bade):

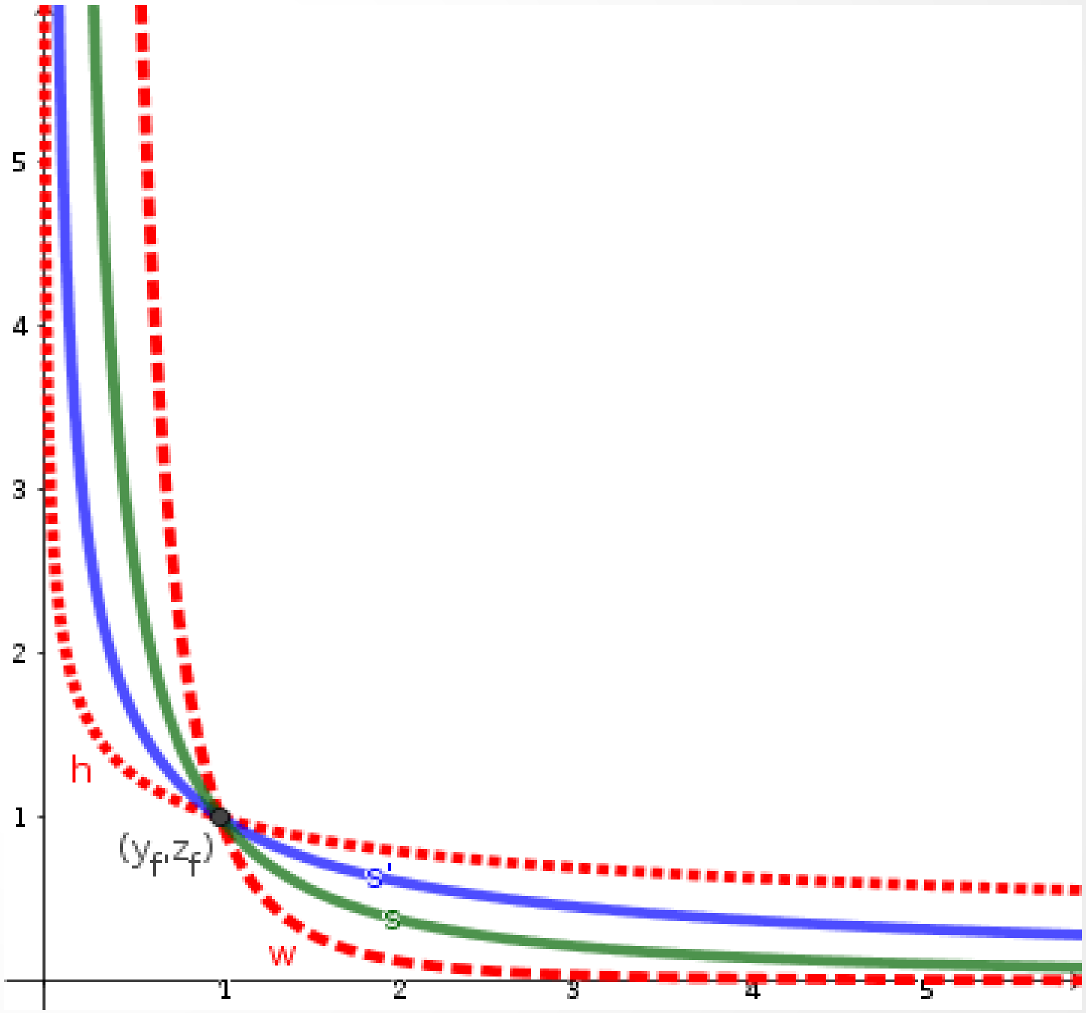
- PO+EF - incompatile for 3 or more families;
  compatible for 2 families.
- PO+FS – always compatible.

2. Homog. div. – families

Theorem 2-: With 3 families, a PO+EF division might not exist.

Proof: 3 families:

- 1 couple, 2 singles.
- Cobb-Douglas prefs.
- EF $\rightarrow$ each single must consume same bundle as family.
- Singles consume same bundle $\rightarrow$ not PO.



2. Homog. div. – families

Theorem 2+: If individuals' preferences are represented by continuous utility functions, then a *Pareto-optimal fair-share* allocation exists.

Proof: Let F := set of all FS allocations.
X := FS allocation that maximizes sum of utilities

- X exists by continuity and compactness of F.
- X is FS since it is in F.
- X is PO since a Pareto-improvement of X would also be in F, contradicing the maximality of X.

2. Homog. div. – families

Theorem 2++: If there are 2 families, and agents‘ preferences are continuous & convex, then a *Pareto-optimal envy-free* allocation exists.

Proof: Let X be a PO+FS allocation.

- X exists by previous theorem.
- X is EF. Suppose member i of family 1 envied family 2. Then i would prefer $(Endowment - X_1)$ over X_1. By convexity, i would prefer $Endowment/2$ to $X_1 \rightarrow$ X were not FS.

3. Indivisible – individuals

Fairness in an economy of individuals:

- *Envy-free-except-c (EFc)*: each individual weakly prefers his share to any other share when some c goods are removed from it.
- *1-of-c maximin-share (MMS)*: each individual weakly prefers his share to dividing the goods into c piles and getting the worst pile.

Theorem (Budish, 2011): for n individuals, an EF1 and 1-out-of-(n+1)-MMS allocation exists.

3. Indivisible – families

Fairness in an economy of families:

- *Envy-free-except-c (EFc)*: each individual weakly prefers his *family*'s share to any other share when some c goods are removed from it.
- *1-of-c maximin-share (MMS)*: each individual weakly prefers his *family*'s share to dividing the goods to c piles and getting the worst pile.

Theorem (with Warut Suksompong): for any finite integer c, even with 2 families, there might be no allocation that is EFc and/or 1-of-c-MMS.

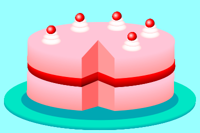
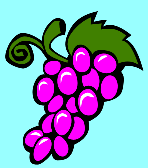
3. Indivisible – families

Theorem 3-: for any finite integer c, there are instances with 2 families, with *binary additive* valuations, where no allocation is EFc and/or 1-of-c-MMS.

Proof: There are 2*c goods. For each distinct subset of c goods, each family has a member who assigns utility 1 to exactly these c goods and utility 0 to the other c goods.

In any allocation, at least one individual has utility 0, so for him, it is not EFc nor 1-of-c-MMS.

Interim Summary

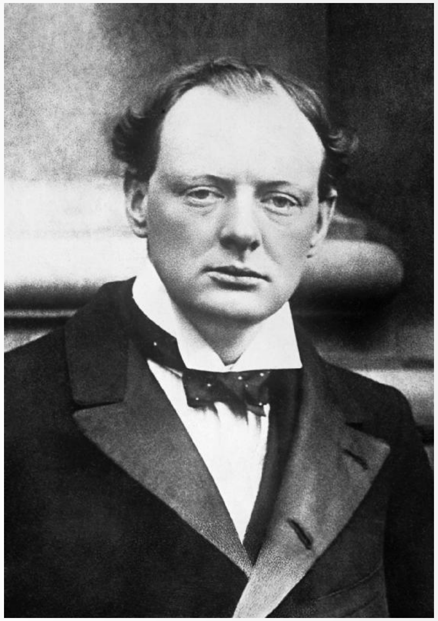
Resource	Challenge	Individuals	Families
1. Het +Div 	EF+CON PR+CON	Yes Yes	No for 2+ families No for 2+ families
2. Hom +Div 	EF+PO FS+PO	Yes Yes	No for 3+ families Yes
3. Indiv 	EFc 1-of-c-MMS	Yes (c=1) Yes (c=n+1)	No for 2+ families No for 2+ families

Unanimous fairness is too much to ask for.

Democratic Fairness

"Democracy is the worst form of government.

...except all the others that have been tried." (Winston Churchil)



Definition: h-democratic fairness ($h \in [0,1]$) := fairness in the eyes of at least a fraction h of the agents in each family.

- *We saw: 1-democratic fairness is impossible.*
- *For what h is h*-democratic fairness possible?

1. Heter. div. – democratic

Theorem 1+: For every integer k, for every k families, there exists a connected $1/k$-democratic EF+PR division.

Proof: Run an existing protocol for finding a connected EF+PR division (Su, 1999).

- Whenever a family has to choose the best of k pieces, let it choose using *plurality voting*.
- At least $1/k$ members of each family are happy with the family's choice.

1. Heter. div. – democratic

Theorem 1+: For every integer k, for every k families, there is a connected $1/k$-democratic EF+PR division.

Corollary: For every 2 families, we can find a connected allocation that will win a (weak) majority in a referendum.

Questions:

- Can we get a support larger than 1/2?
- Can we get a support of 1/2 with 3 families?

1. Heter. div. – democratic

Theorem 1-: For every integer k, there are instances with k families where no connected allocation is more than $1/k$-democratic EF/PR.

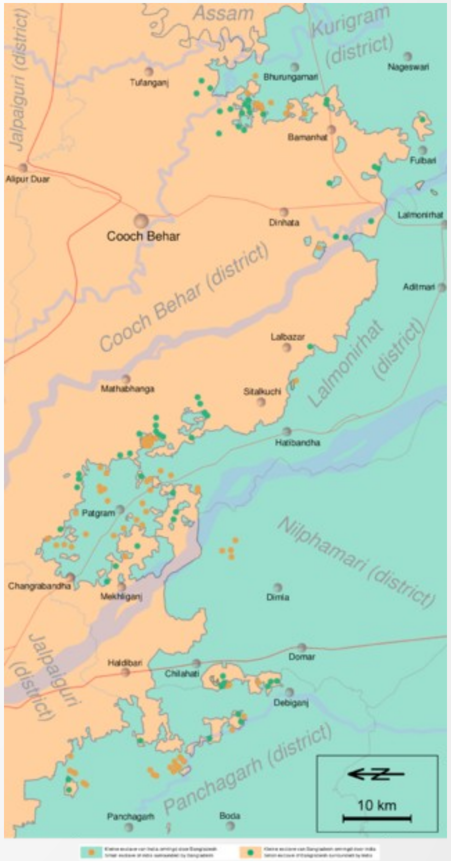
Proof: A family with k members, and k-1 singles. Each individual wants distinct segment:

In any connected division, if two or more members of the family receive non-zero utility, then one single receives zero utility.

1. Heter. div. – democratic

Theorem 1-: For every integer k, there are instances with k families where no connected allocation is more than $1/k$-democratic EF/PR.

- With 2 families: cannot guarantee the support of more than 1/2.
- With 3 or more families: cannot guarantee even a weak majority.



Possible solution: compromise on the *connectivity* requirement.

1. Heter. div. – democratic

Example theorems (proofs in paper):

- For 2 families with n individuals in total:
  There is a 1-democratic EF+PR division with $\boldsymbol{n}$ connected-components;
  There might be no 1-democratic EF+PR division with less than $\boldsymbol{n}$ components.

- For 3 families with n individuals in total:
  There is a 1/2-democratic EF+PR division with $\boldsymbol{n/2+2}$ connected-components;
  There might be no 1/2-democratic EF+PR division with less than $\boldsymbol{n/4}$ components.

1. Heter. div. – democratic

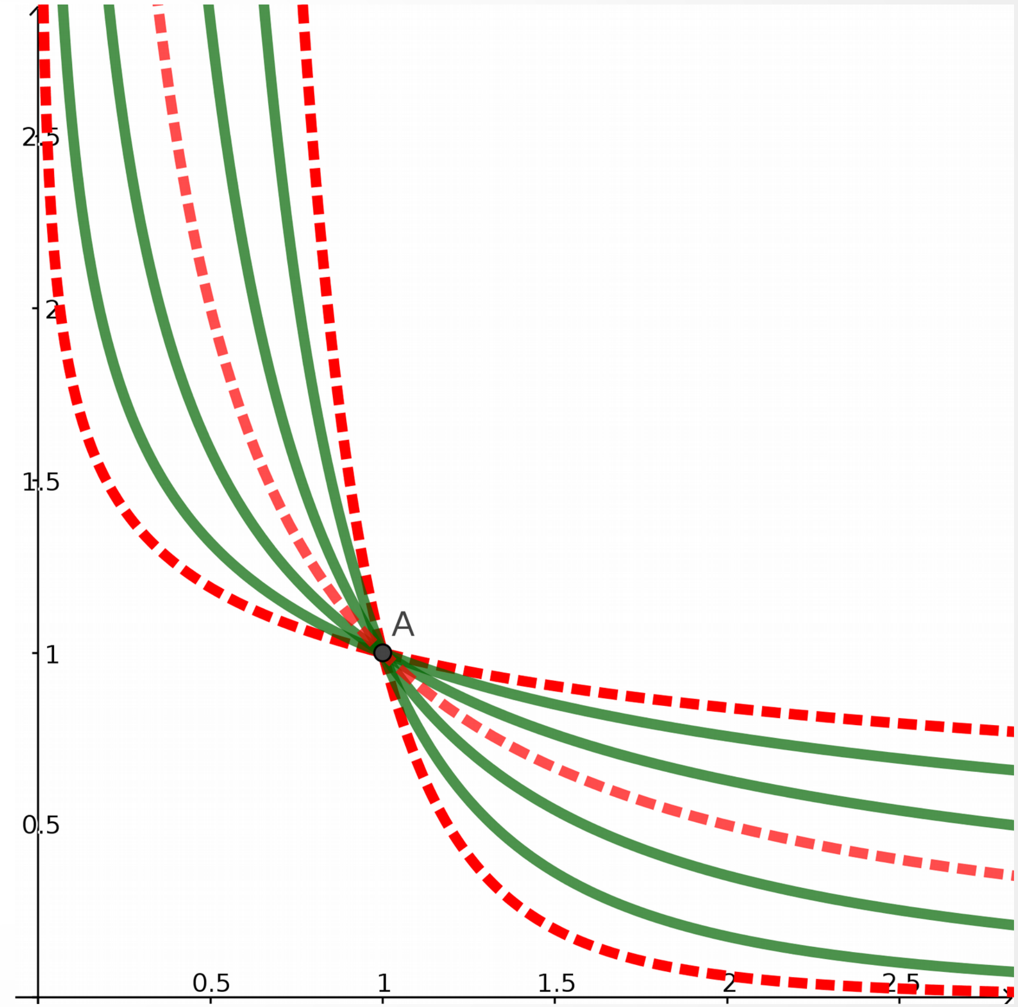
Open questions:

- [Combinatorial] How many components we need for 3 families (between $n/4$ and $n/2+2$)?
  - * Useful for small families only.
- *[Geometric] Can we have a connected fair division of a 2-dimensional resource?*

2. Homog. div. – democratic

Theorem 2-: With **2*k*-1** families, There might be no PO allocation that is EF for more than **1/*k*** the members in each group.

- **Proof**: 2*k*-2 singles + family with *k* members.
- Example for *k* = 3 →
- If at least two members of the family are EF – the allocation is not PO.



2. Homog. div. – democratic

Open question: with **3** or **4** families, is there always a PO allocation that is EF for at least **1/2** the members in each family?

In other words: *can we find a PO allocation that will win a (weak) majority in a referendum?*

3. Indivisible – democratic

Theorem 3+: For every integer k and k families, there is a $1/k$-democratic “EF–2“ allocation.

Proof idea:

- Put all goods on a line.
- Treat the line as a cake.
- Find a connected $1/k$-democratic EF division.
- “Slide“ the cuts to be between the goods.
  * This creates less than 2 “envy units“.

3. Indivisible – democratic

Theorem 3++: For $k=2$ families, there is a $1/2$-democratic EF1 allocation.

Proof: EF1: same as Theorem 3+, but now the cut-sliding creates only 1 “envy unit“.

Corollary: For 2 families, there is an allocation that may win a (weak) majority in a referendum.

- Can we get a support larger than 1/2?
- Can we get a support of 1/2 with 3 families?

3. Indivisible – democratic

Theorem 3-: For every integer k, there are instances with k families with *binary additive* valuations, where no allocation is more than $1/k$-democratic EF1 (proof in paper).

- With 2 families: cannot guarantee the support of more than 1/2.
- With 3 or more families: cannot guarantee even a weak majority.

Possible solution: compromise on the fairness requirement.

3. Indivisible – democratic

Theorem 3++: For every integer $c \geq 1$, for 2 families, when all agents have *binary additive* valuations, there exists a $(1 - 1/2^{c-1})$-democratic 1-out-of-c MMS allocation. *Examples*:

- 1/2-democratic 1-out-of-2 MMS;
- 3/4-democratic 1-out-of-3 MMS;
- 7/8-democratic 1-out-of-4 MMS;

3. Indivisible – democratic

Proof idea: *Round-robin* protocol with *approval voting*.

- Each family in turn picks a good. To decide what to pick, the family uses *weighted approval voting.*
- Each family member is assigned a *potential* based on his number of **r**emaining wanted goods, and the number of goods he **s**hould receive for the fairness.
- The potential of a “winning“ agent increases; the potential of a “losing“ agent decreases.
- The *voting weight* of an agent is his potential-decrease in case he loses.

3. Indivisible – democratic

Potential table for round-robin protocol:

(boldface cells correspond to 1-of-3-MMS)

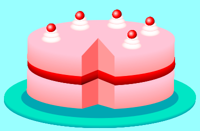
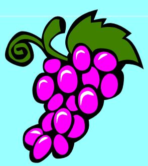
$r\downarrow \quad s\rightarrow$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	**1**	0	0	0	0	0	0	0	0	0
2	**1**	0.5	0	0	0	0	0	0	0	0
3	1	**0.75**	0	0	0	0	0	0	0	0
4	1	**0.875**	0.375	0	0	0	0	0	0	0
5	1	**0.938**	0.625	0	0	0	0	0	0	0
6	1	0.969	**0.782**	0.313	0	0	0	0	0	0
7	1	0.985	**0.876**	0.548	0	0	0	0	0	0
8	1	0.993	**0.931**	0.712	0.274	0	0	0	0	0
9	1	0.997	0.962	**0.822**	0.493	0	0	0	0	0
10	1	0.999	0.98	**0.892**	0.658	0.247	0	0	0	0
11	1	1	0.99	**0.936**	0.775	0.453	0	0	0	0

3. Indivisible – democratic

Proof idea (cont.):

- Potentials are calculated such that, for each agent:
  the potential increase in case he loses
  ≥ the potential decrease in case he loses
- Hence, the total family potential always increases.
- At the end, the potential is:
  1 for an agent who feels the division is fair;
  0 for an agent who feels it is unfair.
- The fraction of happy agents is at least the smallest initial potential of an agent, which is $\left(1 - 1/2^{c-1}\right)$.

Summary

Resource	Challenge	h-democratic
1. Het +Div 	EF+CON PR+CON	k families: $h = 1/k$. k families: $h = 1/k$.
2. Hom +Div 	EF+PO FS+PO	2k-1 families: $h \leq 1/k$. k families: $h = 1$.
3. Indiv 	EF2 / EF1 1-of-c-MMS	k families: $h = 1/k$. 2 binary families: $1 - 1/2^{c-1}$

Conclusion

When dividing goods among families:

- Unanimous fairness is usually impossible.
- 1/2-democratic fairness is often possible for the common case of **2 families**.

Main open questions for **3 families**:

- **Het+div:** #components for envy-free?
- **Hom+div**: PO 1/2-democratic envy-free?
- **Indiv**: 1/2-democratic 1-of-c-MMS?

Thank you!