Fractional and integral matchings in d-partite hypergraphs

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Based on joint work (work in progress) with:

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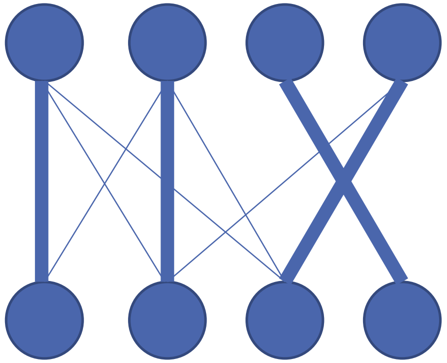
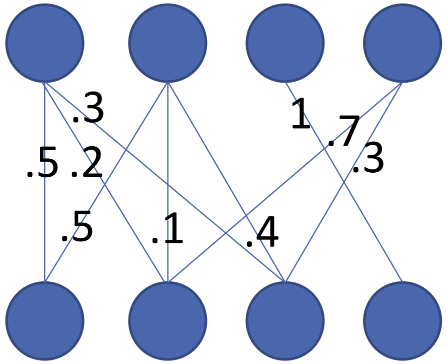
(n,n) bipartite graphs

Perfect matching



Set of edges; each vertex is contained in exactly one edge.

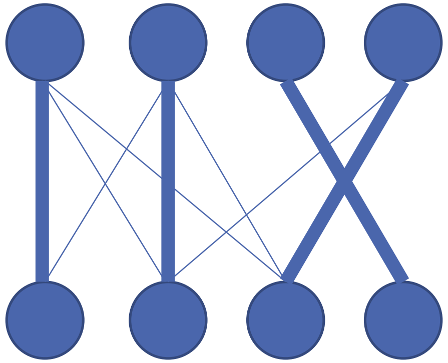
Perfect fractional matching



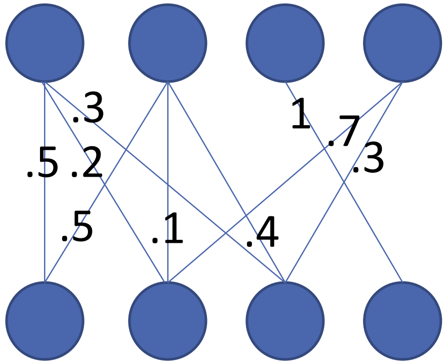
Non-negative weight function on edges; total weight near each vertex = 1. *(equivalently: total weight near each vertex is constant)*.

(n,n) bipartite graphs

Perfect matching



Perfect fractional matching



Often, we can prove the existence of a *fractional* matching,
but we need an (integral) matching.

Koenig (1916): Perfect fractional matching $\longrightarrow$ Perfect matching.

Example application: fair cake-cutting

n agents with
different preferences:

Preference = set of most-wanted
pieces in each n-partition:

Cake (interval) should
be partitioned into n intervals:

Meunier & Su (2019): For any n “hungry agents”, there is an n-partition in which agent-piece graph has a *balanced* weight-function (= constant vertex-weights).

+ Koenig (1916): exists n-partition with envy-free allocation of pieces to agents.

Our goal: extend Koenig’s theorem to ***d*-partite hypergraphs.**

Focus: **tripartite hypergraphs.**

(n, n, n) tripartite hypergraphs

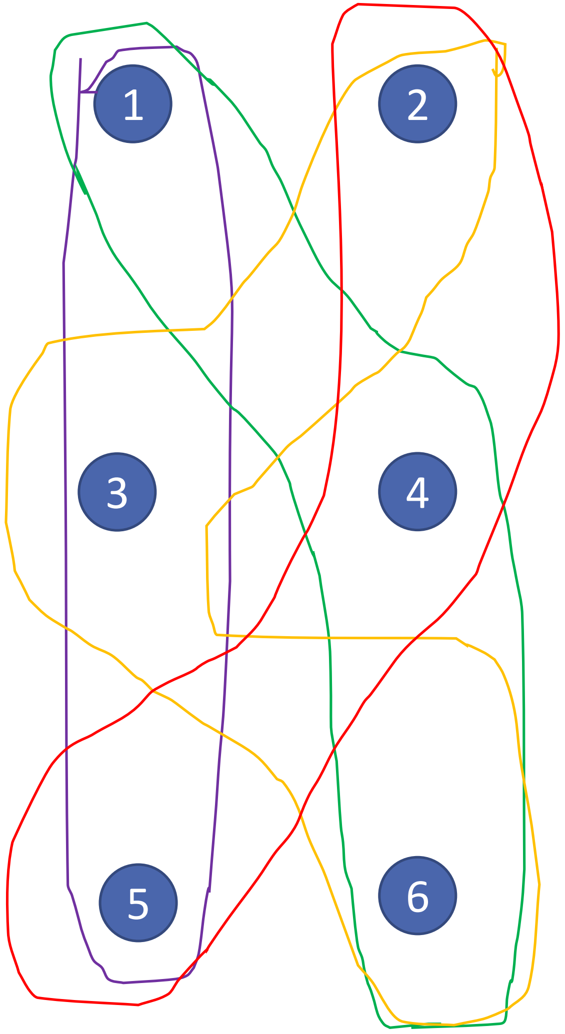
Furedi (1981): Fractional matching of size n

$\rightarrow$ integral matching of size ceil$(n/2)$.

It is tight.

Example for n=2:

{(1,3,5), (1,4,6), (2,3,6), (2,4,5)}



Example application: fair *multi*-cake-cutting

n hungry agents with
different preferences:

Preference = set of most-wanted
piece-pairs in each *pair* of partitions:

Each cake should be partitioned
into n intervals:

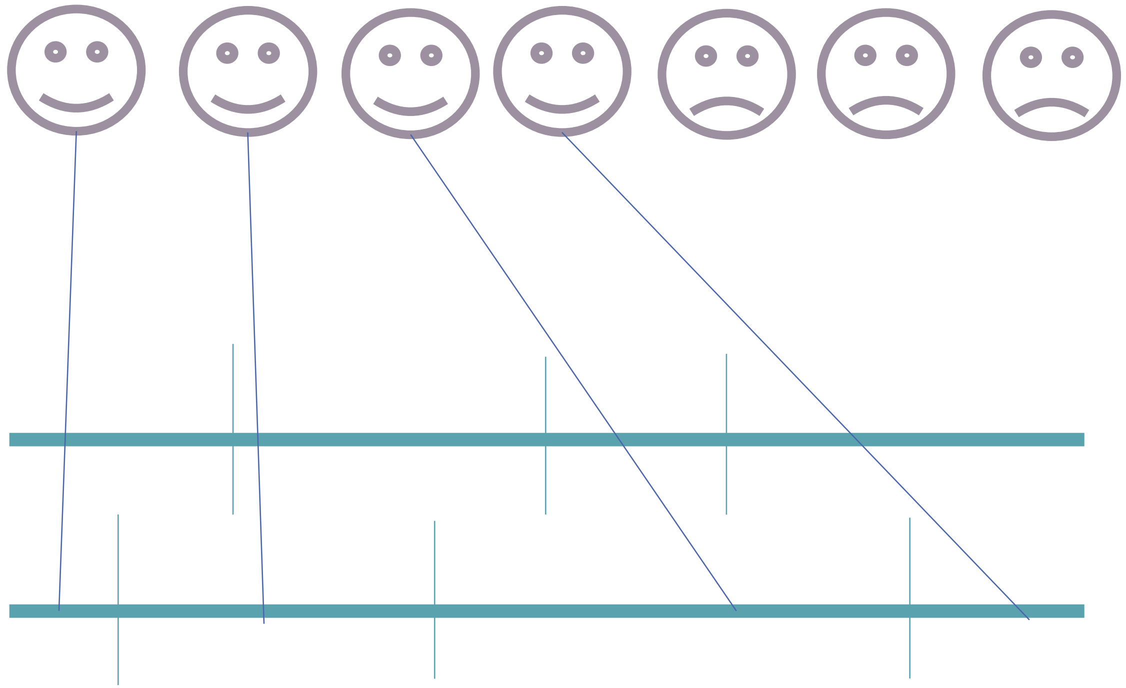
Theorem: There exists a pair of n-partitions with a balanced weight function.

+ Furedi (1981): There exists an envy-free allocation of $n/2$ pairs to $n/2$ agents.

Example application: fair *multi*-cake-cutting

$2n$-1 hungry agents with different preferences:

Each cake should be partitioned into n intervals:

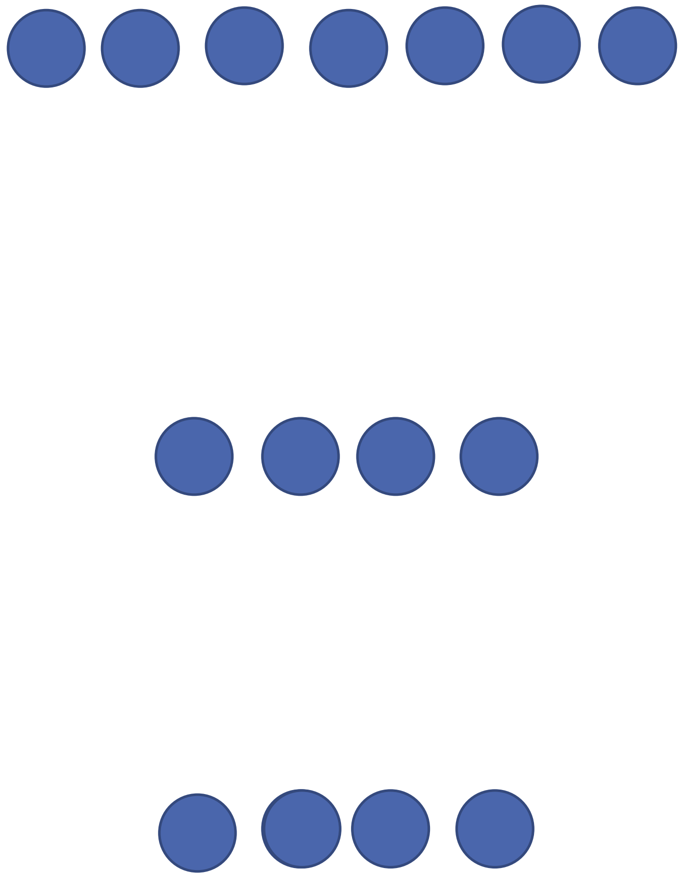


Nyman, Su, Zerbib (2020): For $2n$-1 hungry agents, there exists a pair of n-partitions that allows an envy-free allocation of n pairs to n agents.

(their proof uses a different technique).

Back to tripartite hypergraphs

Conjecture: In any $(n, n, 2n\text{-}1)$ tripartite hypergraph, if there exists a balanced weight-function, then there exists a matching of size n.



If true, it would imply that we can get an envy-free allocation to n agents, by cutting one cake into n pieces and the other into $2n$-1 pieces.
(donating n-1 pieces to charity).

We refuted it, but proved weaker variants, e.g.:
for $(n, n, n^2\text{-}n/2)$ and $(n, 2n\text{-}1, 2n\text{-}1)$

Balanced weight functions

Definition: In a d-partite hypergraph,
a weight function is called *balanced* if in each side,
the total weight near each vertex is a constant.

Notation: $\mathrm{BM}(n_1,n_2,n_3) :=$
Largest m such that every (n_1,n_2,n_3)-tripartite hypergraph with a balanced weight function has a matching of size m.

Koenig (1916): $\mathrm{BM}(n,n) = n$

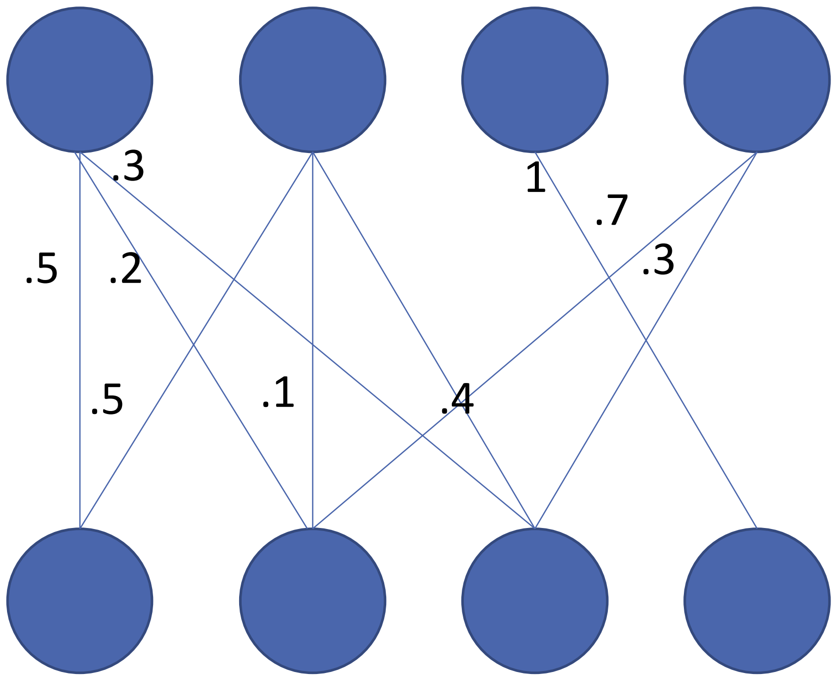
Furedi (1981): $\mathrm{BM}(n,n,n) = \mathrm{ceil}(n/2)$

Our goal: calculate $\mathrm{BM}(n_1,n_2,n_3)$ for different n_1,n_2,n_3.

Proof Technique

- Koenig's theorem $\mathrm{BM}(n,n)=n$ can be proved using Hall's theorem for bipartite graphs.

- We prove lower bounds on $\mathrm{BM}(n_1,n_2,n_3)$ using a Hall-type theorem for bipartite hypergraphs.

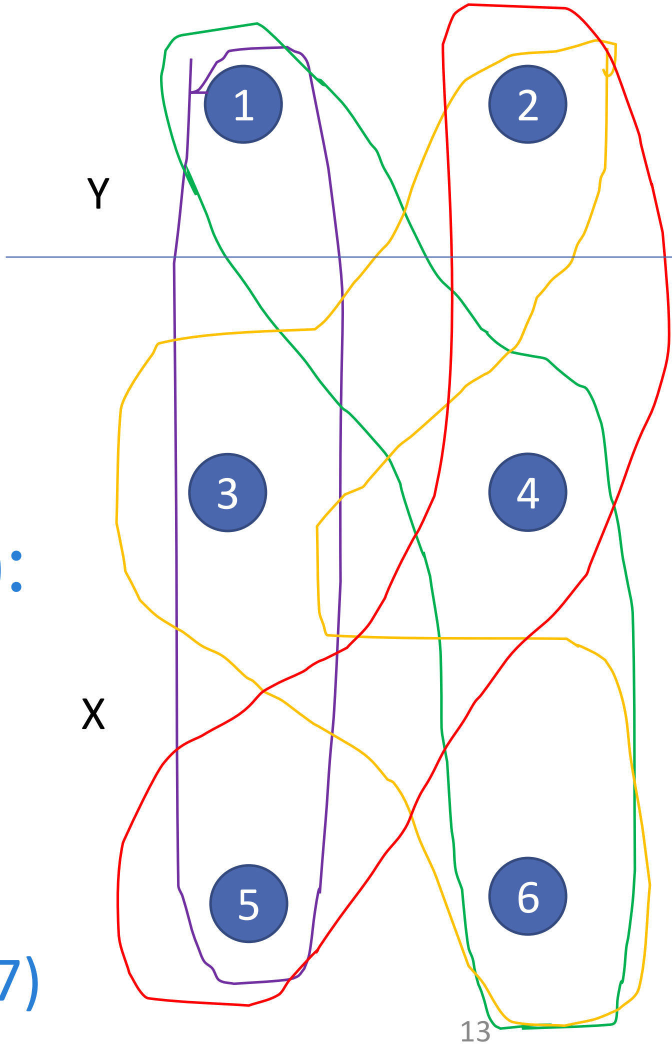


Hall-type theorems for bipartite hypergraphs

Bipartite hypergraph: vertices are partitioned into X, Y; each edge contains exactly one vertex of Y.

Neighbor set: $\mathrm{N}(Y') := \{X' \subseteq X \mid \{y'\} \cup X' \text{ is an edge for some } y' \text{ in } Y'\}$.

Example: $\mathrm{N}(\{1\}) = \{ \{3,5\}, \{4,6\} \}$.



Hall's theorem considers the *size* of $\mathrm{N}(Y')$ vs. $|Y'|$.

Its generalizations consider other properties of $\mathrm{N}(Y')$:

- Matching number of $\mathrm{N}(Y')$ (Aharoni & Kessler, 1990)
- Covering number of $\mathrm{N}(Y')$ (Haxell, 1995)
- Matching width of $\mathrm{N}(Y')$ (Aharoni & Haxell, 2000)
- $\Psi(\mathrm{L}(\mathrm{N}(Y')))$ (Meshulam, 2003; Aharoni&Berger&Ziv, 2007)

Meshulam's game

A two-player zero-sum turn-based game on a given graph G:

- Player 1 ("CON") picks an edge e.
- Player 2 ("NON") has two options:
  - **Disconnect** – remove only e.
  - **Explode** – remove e, its two endpoints, and their neighbors; this action requires NON to pay 1 point to CON.

The game ends when no edges remain:

- If no vertices remain, then CON's score is the num of points;
- If some isolated vertices remain, then CON's score is infinite.

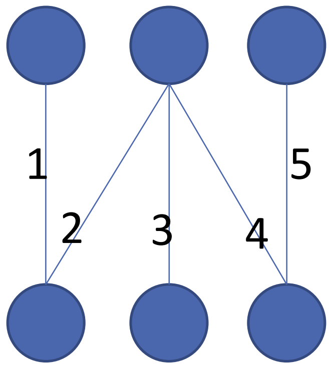
$\Psi(G)$:= score of CON when both players play optimally on G.

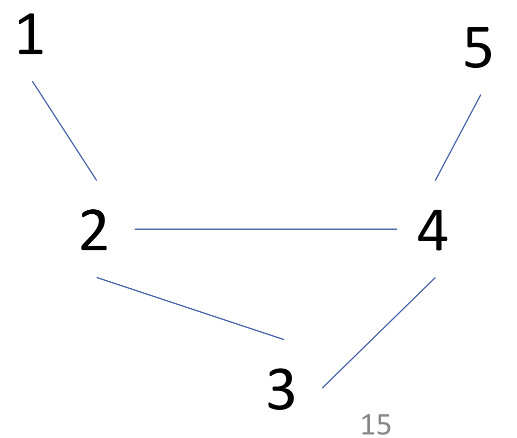
Meshulam's game

Simple examples:

- If G has k connected components, then $\Psi(G) \geq k$.
- If G is the union of k disjoint cliques, then $\Psi(G) = k$.

The *line-graph* of G is denoted $\mathrm{L}(G)$:





Hall-type theorem (Meshulam 2003,2004):

Given a bipartite hypergraph with sides X, Y:

If for every $Y' \subseteq Y$: $\Psi(\mathrm{L}(\mathrm{N}(Y'))) \geq |Y'|$,

then there is a matching of size $|Y|$.

Remark. In a bipartite graph, $\Psi(\mathrm{L}(\mathrm{N}(Y'))) = |\mathrm{N}(Y')|$.

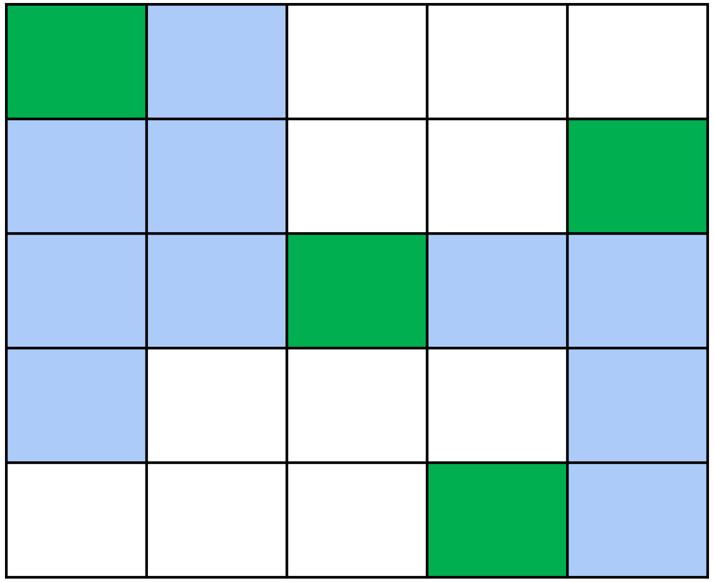
Meshulam's game on $\mathrm{L}(G)$ of bipartite graphs

Lemma: If G is a bipartite graph with a matching of size m,
then $\Psi(\mathrm{L}(G)) \geq \mathrm{ceil}(m/2)$.

Proof sketch: G = array of cells:
row / column = vertex on one / other side;
cell = possible edge (green cell = edge of matching).

- Cells of G are vertices in $\mathrm{L}(G)$;
  Pairs in same row/column are edges in $\mathrm{L}(G)$.
- Each explosion destroys one row and two cols or one col and two rows.
- CON offers pairs with a green cell; each explosion destroys at most 2 green cells ***

$m = 4$



Back to matchings in tripartite hypergraphs

Theorem: For every $n \leq k \leq 2n$: $\mathrm{BM}(n, k, k) \geq \mathrm{ceil}(k/2)$.

(in words: every (n, k, k)-tripartite hypergraph
with a balanced weight function has a matching of size $\mathrm{ceil}(k/2)$).

Corollary: $\mathrm{BM}(n, 2n\text{-}1, 2n\text{-}1) = n$.

Proof idea: Given a (n, k, k)-tripartite hypergraph H, let

- Y = the side of size n;
- X = the other two sides.

For every $Y' \subseteq Y$, the set $\mathrm{N}(Y')$ is a bipartite graph on X.

The balanced weight function on H induces
a fractional matching on $\mathrm{N}(Y')$ with total weight $|Y'|$.

By Koenig's theorem, $\mathrm{N}(Y')$ has a matching of size $|Y'|$.

By previous lemma, $\Psi(\mathrm{L}(\mathrm{N}(Y'))) \geq \mathrm{ceil}(|Y'|/2)$.

Meshulam's game with balanced weight functions

Lemma: If G is a (n,rn)-bipartite graph with balanced weight func.,

then $\Psi(\mathrm{L}(G)) \geq \mathrm{ceil}(rn/(r+1))$.

Proof sketch (for special case: $r \geq 1$ is an integer):

Step 1. an (n,rn)-bipartite graph with a balanced weight function has a 1-to-r matching of size rn.

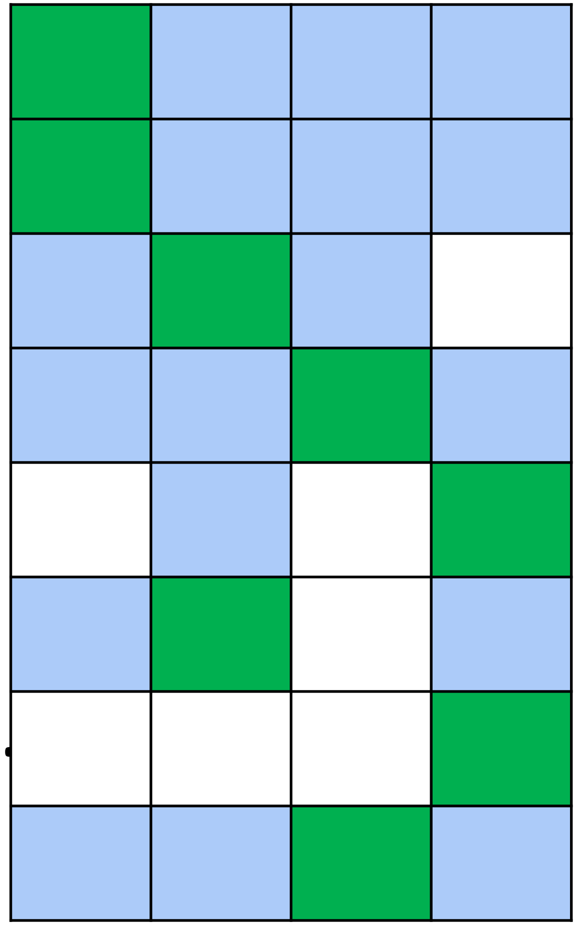
Step 2. Play Meshulam's game on this graph: CON can play such that each explosion destroys at most $(r+1)$ cells of matching.

Theorem: For $n, r \geq 1$: $\mathrm{BM}(n, n, rn) \geq \mathrm{ceil}(rn/(r+1))$.

Corollaries: $\mathrm{BM}(n, n, n^2) = n$

$\mathrm{BM}(n, n, 2n\text{-}1) \geq \mathrm{ceil}((2n\text{-}1)/3)$

$n = 4, r = 2$



Upper bounds

For $n < k$: $\mathrm{BM}(n, k, k) \leq \text{floor}((3k+1)/4)$

If n-ceil(k/2) divides ceil(k/2): $\mathrm{BM}(n, k, k) \leq \text{ceil}(k/2)$

[Recall lower bound: for every $n \leq k \leq 2n$: $\mathrm{BM}(n, k, k) \geq \mathbf{ceil}(\boldsymbol{k}/\mathbf{2})$]

For $n, r \geq 1$: $\mathrm{BM}(n, n, rn) \leq 2rn\ /\ (2r+1)$

[Recall lower bound: for $n, r \geq 1$: $\mathrm{BM}(n, n, rn) \geq \text{ceil}(rn/(r+1))$.]

Thank you!