"DIVIDE THE LAND EQUALLY" (Ezekiel 47:14)

Fair Division with Minimal Sharing

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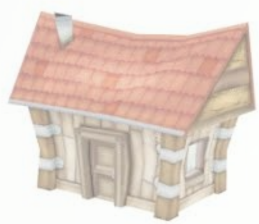
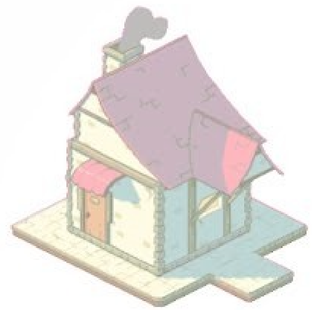
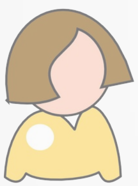
Setting

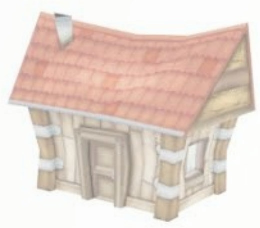
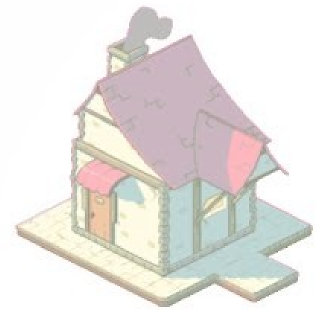
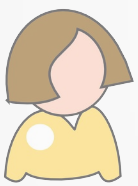
Input: m objects, n agents, additive valuations:

Object:					
 v_A:	3	2	3	1	2
 v_B:	1	1	1	5	4

Goal: Envy-Free and Pareto-Optimal division.
No monetary transfers.

Main problem

The objects are **indivisible**:

Object:					
 v_A:	3	2	3	1	2
 v_B:	1	1	1	5	4

→ An envy-free Pareto-optimal allocation might not exist.

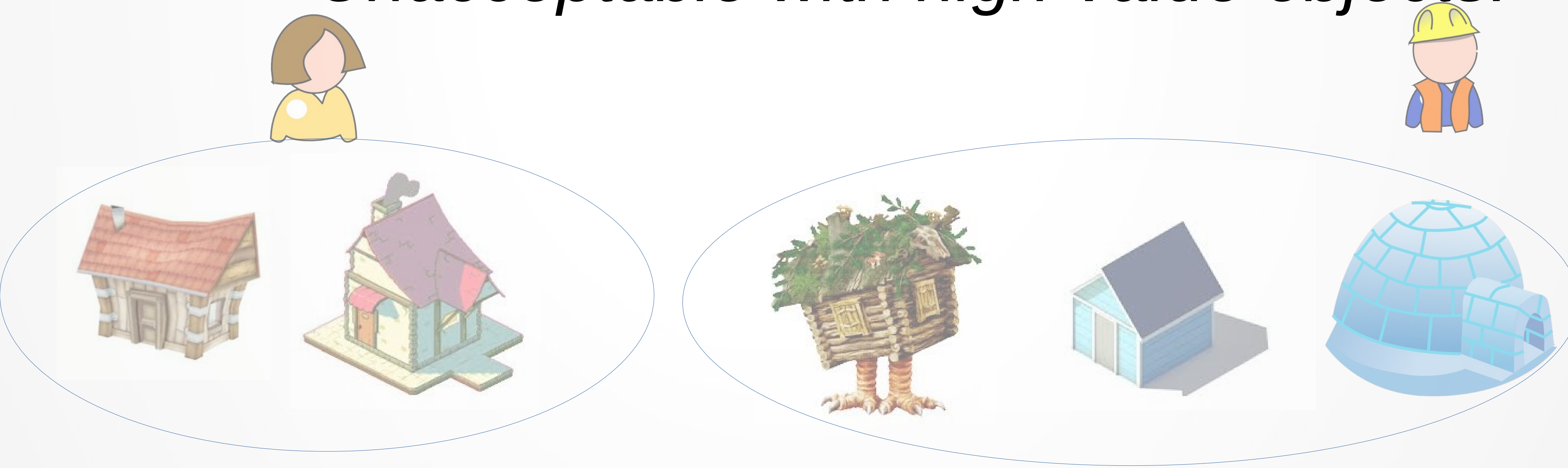
Handling indivisible objects

Common approach in computer science:

Approximate fairness, e.g.:

Envy-free except one object (EF1)

– *Unacceptable with high-value objects.*



Handling indivisible objects

Common approach in economics:

Randomization:

Envy-free in expectation.

– *Unacceptable with high-value objects.*

Handling indivisible objects

Our approach:

Minimal Sharing:

Find the *smallest* number of objects that must be shared to attain PO+EF.

Problem statement

INPUT: valuation matrix $\mathbf{v} : n \times m$.

$v_{i,o}$ = value of object o to agent i.

[Initially we assume: $v_{i,o} > 0$]

OUTPUT: allocation matrix $\mathbf{z} : n \times m$.

$z_{i,o}$ = fraction of object o given to agent i.

[Everything is allocated: $\sum_i z_{i,o} = 1$ for all o]

[Additive utilities: $u_i(\mathbf{z}) := \sum_o z_{i,o} v_{i,o}$]

GOAL: minimize #sharings($\mathbf{z}$) s.t. $\mathbf{z}$ is PO+EF.

#sharings(z) := $\# \{ (i,o) \mid z_{i,o} > 0 \} - m$

[#sharings(z) $= 0$ iff no objects are shared]

Solution outline

Step 1. Upper Bound:
A PO+EF allocation with at most n-1 sharings always exists and can be found efficiently.

Step 2. Minimization Algorithm:
A PO+EF allocation with a minimum number of sharings can be found in time polynomial in m, if n is fixed.

Step 1: Upper Bound

Preliminary Theorem.

A PO+EF allocation with $\leq n$-1 sharings exists and can be found in time O(poly(m,n)).

Step 1: Upper Bound

Preliminary Theorem.
A PO+EF allocation with $\leq n$-1 sharings exists and can be found in time $O(m^2n^2(m+n))$.

Proof Sketch.

- (a) Ignoring the number of sharings, find an allocation maximizing the *product* of utilities.
- (b) Modify the allocation to have $\leq n$-1 sharings, without changing utilities.
- (c) The allocation still maximizes the product. Any such allocation is PO+EF.

Step 1: Upper Bound

1(a): Find an allocation z maximizing the product of utilities, ignoring #sharings:

– Can be done in time $O((n+m)^4 \log(n+m))$.

James B Orlin. "Improved algorithms for computing Fisher's market clearing prices" (STOC 2010).

– Any such allocation is PO+EF.

E Eisenberg and D Gale, "Consensus of subjective probabilities" (Annals of Math. Statistics, 1959).

Step 1: Upper Bound

1(b): Find a new allocation z^* with same utilities ($U_1,\ldots,U_n$) and ≤ n-1 sharings.

LP-based proof *(extends Stephen Wilson, "Fair division using linear programming", preprint, Iowa State University, 1998):*

$$\text{maximize} \quad \sum_i \sum_o z_{i,o} v_{i,o} \quad \text{(sum of utilities)}$$

$$\text{subject to} \quad \sum_i z_{i,o} = 1 \quad \forall o \in [m] \quad \text{(feasibility)}$$

$$\sum_o z_{i,o} v_{i,o} = U_i \quad \forall i \in [n-1] \quad \text{(same utility)}$$

Utility of n-th agent must be U_n too.

m+n-1 constraints → Basic sol. has ≤ m+n-1 nonzeros. → Allocation has ≤ n-1 sharings.

Step 1: Upper Bound

1(b): Find a new allocation $\mathbf{z}^*$ with same utilities ($U_1,\ldots,U_n$) and $\leq n$-1 sharings.

Polytime proof (extends "Dividing goods or bads under additive utilities", Bogomolnaia&Moulin&Sandomirskiy&Yanovskaya, 2016).

- Construct the directed consumption graph of $\mathbf{z}$.
- $\mathbf{z}$ is PO $\rightarrow$ no cycles with weight-product < 1.
- Remove each cycle with weight-product = 1 by trading along the cycle without changing the utilities.
- *Claim*: in the resulting $\mathbf{z}^*$, the *undirected* consumption graph is acyclic (since each such cycle corresponds to two opposite directed cycles with product=1).
- Acyclic graph $\rightarrow$ $\leq m+n-1$ edges $\rightarrow$ $\leq n-1$ sharings.

Step 1: Upper Bound

1(c): The new allocation $\mathbf{z}^*$ has:

- At most n-1 sharings;
- Product-maximizing utilities $U_1,\ldots,U_n$.

→ **It is still PO+EF.**

Conclusion:

- n-1 is an upper bound on #sharings.
- In worst case, n-1 sharings are necessary.
- In some cases, less sharings are sufficient.
- *Next goal: minimize #sharings.*

Step 2: Minimization – 2 agents

Discouraging result:

n=2 agents with *identical* valuations:

- n–1 = 1, so either *0* or *1* sharings.
- Any allocation is PO.
- An allocation is EF iff both agents get exactly the same utility.
- → Equivalent to NP-hard problem **Partition**!

Conclusion: Minimizing the number of sharings is computationally-hard **even** for 2 agents with identical valuations.

Step 2: Minimization – 2 agents

Encouraging result.

n=2 agents with *generic* valuations, i.e.:
the *m* ratios $v_{a,o} / v_{b,o}$ are all different.

- Order the objects by decreasing ratio:
$v_{a,1}/v_{b,1} > v_{a,2} / v_{b,2} > \ldots > v_{a,o} / v_{b,o} > \ldots > v_{a,m} / v_{b,m}$

- **Order Lemma.** In a PO allocation, If Alice gets a positive amount of some object o, then Alice gets *all* objects $o' < o$.
Proof. Otherwise Alice could trade o for o'.

Step 2: Minimization – 2 agents

Encouraging result (cont.).

Any PO allocation must have the form:

A:	**1**	**1**	**…**	***x***	…	0
B:	0	0	…	**1-*x***	**…**	**1**

for some object o and x in [0,1].

- Only $m+1$ PO allocations with 0 sharings.
- It is easy to check each of them for EF.

→ Minimization can be solved in $O(m \log m)$.

Conclusion: For 2 agents, minimizing the number of sharings is computationally-hard ~~even~~ **only** with non-generic preferences.

Step 2: Minimization – 2 agents

Partially-generic valuations.

- *Degree of degeneracy* := smallest d such that for any ratio $r > 0$ there are at most d objects o with $v_{a,o} / v_{b,o} = r$ (generic: $d=1$; identical: $d=m$).
- At most $2^d * m$ allocations with no sharing.

- If $d = O(\log(m))$, the minimization can still be solved in time $O(\text{poly}(m))$.

- If $d = \Omega(m^a)$ for some $a>0$, the minimization is NP-hard (reduction from Partition).

Step 2: Minimization – n agents

Generic valuations:= for every two agents i,j:

the m ratios $v_{i,o} / v_{j,o}$ are all different.

Main Theorem. When n is fixed and the valuations are generic, a PO+EF allocation minimizing the number of sharings can be found in time $O(\text{poly}(m))$.

Step 2: Minimal Sharing

Main Theorem. When n is fixed and the valuations are generic, a PO+EF allocation minimizing the number of sharings can be found in time $O(\text{poly}(m))$.

Proof Sketch.

(a) Define *consumption-graph* of an allocation.

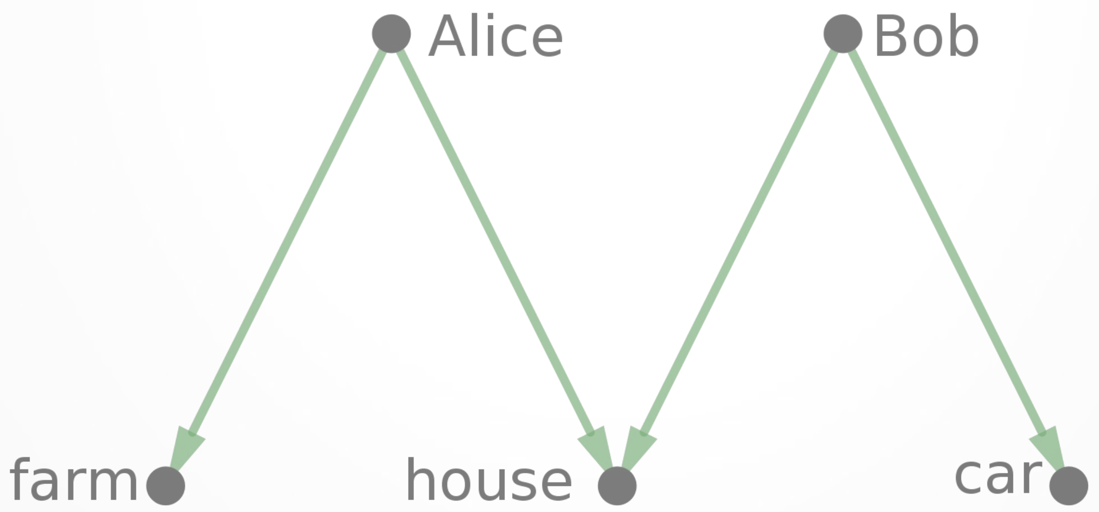
(b) At most $O(m^{n(n-1)/2})$ consumption-graphs correspond to PO allocations, and they can be enumerated with breadth-first search.

(c) Given a consumption-graph, PO+EF can be decided in time $O(m^2)$.

Step 2(a): Consumption graph

Consumption graph of allocation z :=

- Bipartite graph: agents vs. objects.
- Edge between i and o iff $z_{i,o} > 0$.
- Example:

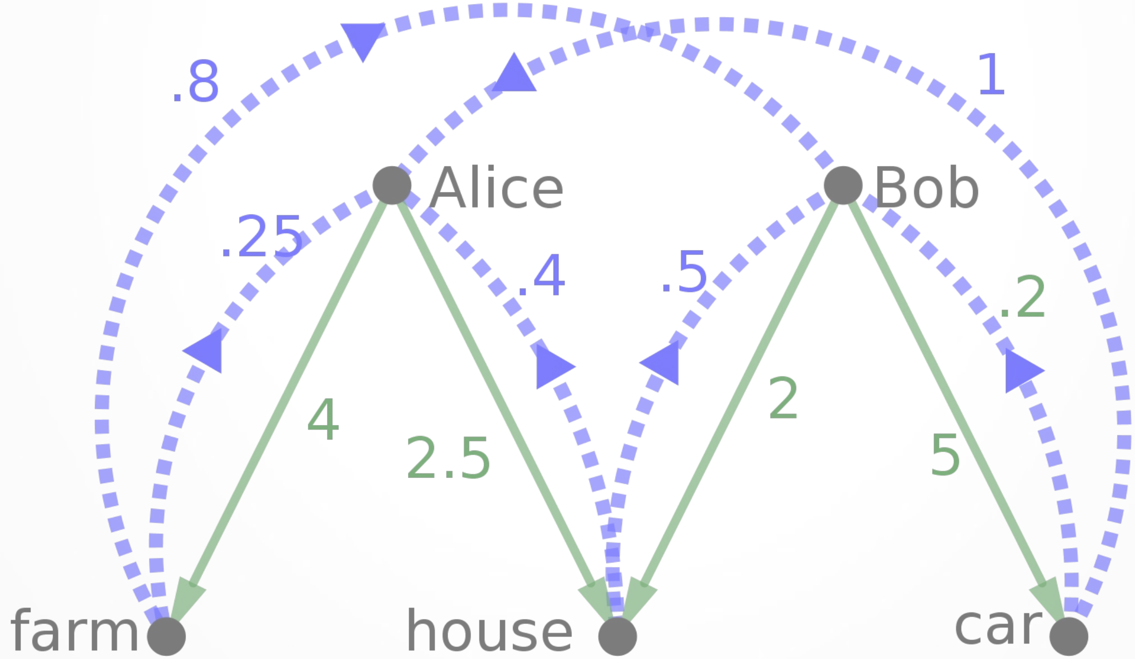


Alice takes farm; Bob takes car; shared house.

Step 2(a): Consumption graph

Directed **consumption graph of** z :=

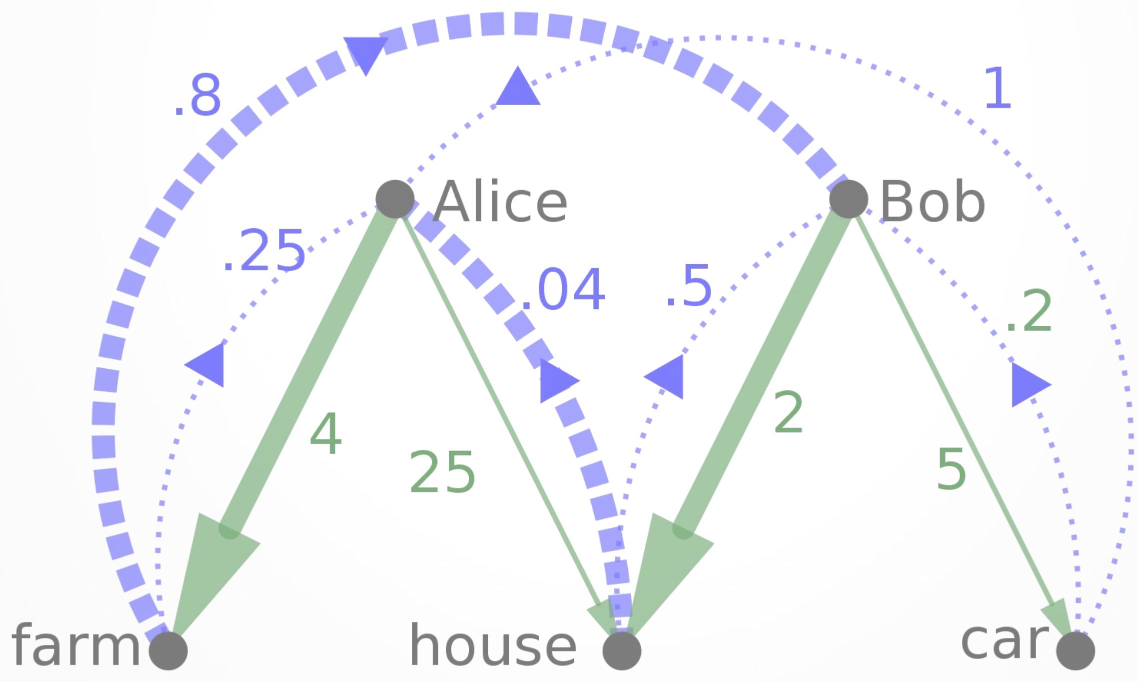
- Edge from i to o iff $z_{i,o} > 0$; weight $v_{i,o}$.
- Edge from o to i ; weight $1/v_{i,o}$.



$v_a = [4, 2.5, 1];$ $v_b = [1.25, 2, 5].$

Step 2(a): Consumption graph

Cycle Lemma. An allocation is PO iff its directed consumption graph contains *no cycles* whose product of weights is < 1 *(like this:)*



$v_a = [4, 25., 1];$ $v_b = [1.25, 2, 5].$

Step 2(a): Consumption graph

Cycle Lemma - *Proof Sketch*.

- $\rightarrow$ Given a cycle with weight-product < 1, we can trade goods among agents in the cycle such that all participants strictly gain.
- $\leftarrow$ Suppose there are no such cycles. Pick an arbitrary agent i with nonempty bundle. Find a minimum-product path from i to every other agent j. Define the *weight* of j as the product of this path. We can prove that the allocation maximizes the *weighted-sum* of the agents' utilities. Hence it is PO.

Step 2(a): Consumption graph

Corollary. It is possible to check whether a consumption-graph corresponds to a PO allocation in time $O(m\, n\, (m+n))$.

Proof.

- Convert weights to their logarithms;
- Use algorithms for negative cycle detection (e.g. Bellman-Ford algorithm).

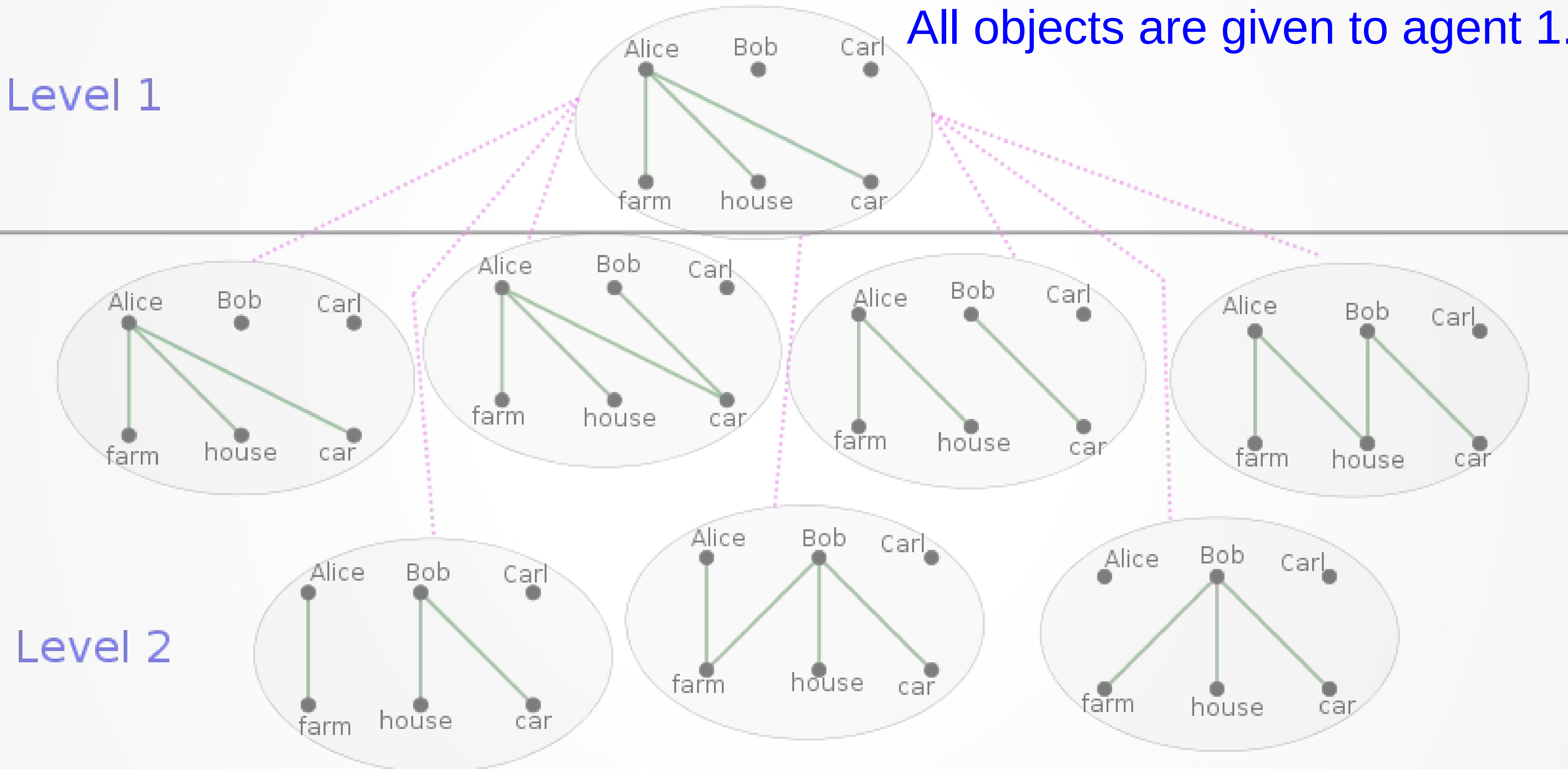
Step 2(b): Tree of PO graphs

2(b). Enumerating PO consumption graphs

- We construct a tree of consumption graphs.
- For all k in $1,\ldots,n$,
  level k will contain the consumption graphs of all PO allocations among agents $1,\ldots,k$.

Step 2(b): Tree of PO graphs

All objects are given to agent 1.



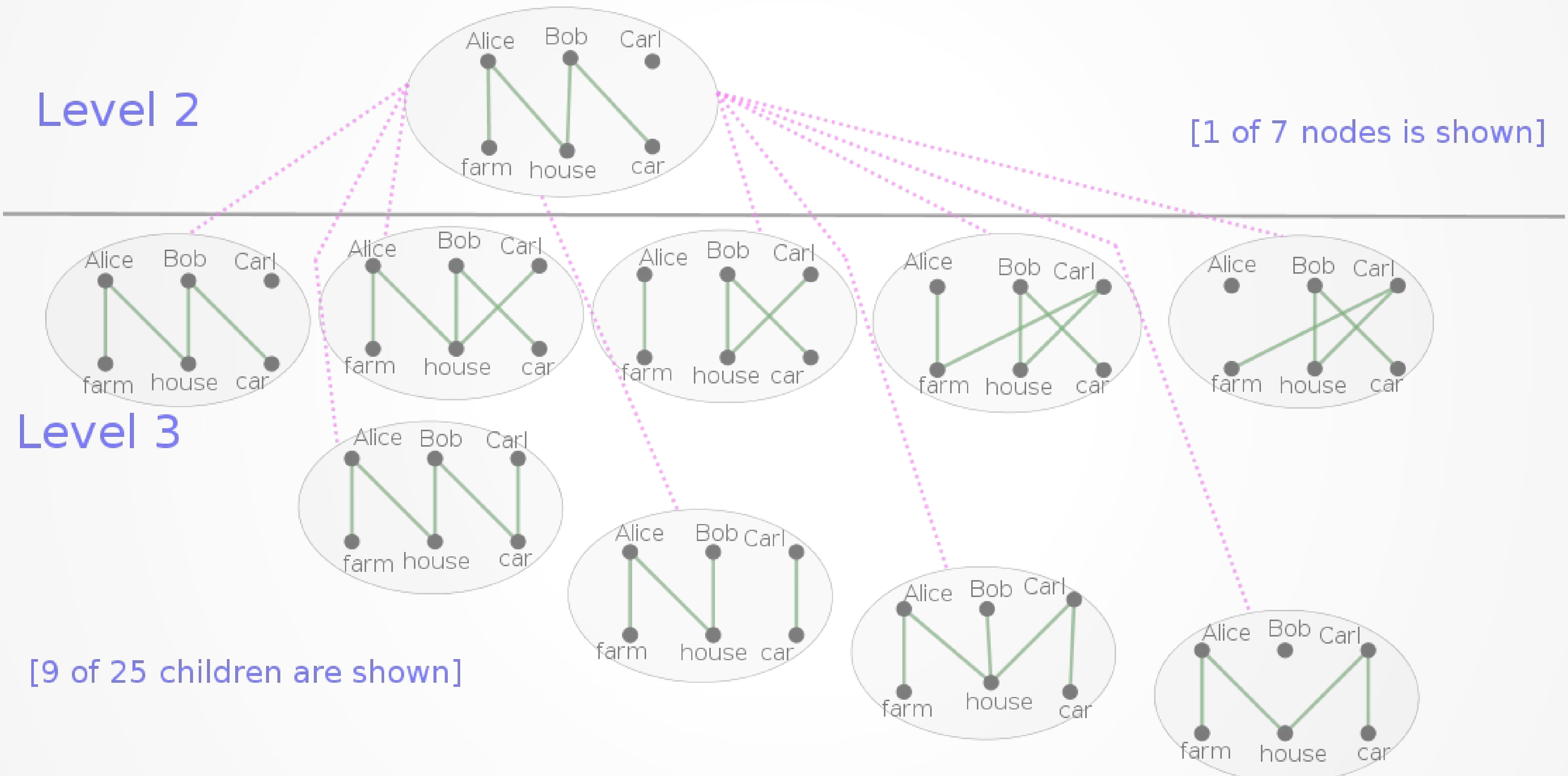
All objects are given to agents 1,2. $2m+1$ options (by Order Lemma).

Step 2(b): Tree of PO graphs

2(b). Enumerating PO consumption graphs

- … In level 3, all objects are given to 1,2,3.
- For each node in level 2:
  - All objects of 1 are divided between 1 and 3;
    - $2m_1+1$ options (m_1 = #objects given to 1).
  - All objects of 2 are divided between 2 and 3.
    - $2m_2+1$ options (m_2= #objects given to 2).
  - $(2m_1+1)(2m_2+1) = O(m^2)$ children per node.
- All in all: $O(m^3)$ nodes in level 3.

Step 2(b): Tree of PO graphs



Step 2(b): Tree of PO graphs

2(b). Enumerating PO consumption graphs

- … In level k, all objects are given to $1,\ldots,k$.:
  - $O(m^{k-1})$ children per node in level k-1.
  - $O(m^{k(k-1)/2})$ nodes in level k.
- All in all: $O(m^{n(n-1)/2})$ leaves (in level n).

- Apply the Cycle Lemma to the leaves.
  - *Only* graphs of PO allocations remain.

- By the Generation Lemma (*next slide*):
  - *All* graphs of PO allocations are in the tree →

Step 2(b): Tree of PO graphs

Generation Lemma. Every PO allocation $\mathbf{z}^{+}$ among agents $1,\ldots,k+1$ can be generated by:

- Taking some PO allocation $\mathbf{z}$ among $1,\ldots,k$;
- Having each agent i in $1,\ldots,k$ share zero or more objects with agent $k+1$.

Proof sketch.

(I) Initialize $\mathbf{z} := \mathbf{z}^{+}$ without the $(k+1)$-th bundle.

(II) Take an object o from the $(k+1)$-th bundle. For each agent i in $1,\ldots,k$, let $\mathbf{z}'(i) := \mathbf{z}$ with o given to agent i. One of these $\mathbf{z}'(i)$ must be PO.*

(III) Let $\mathbf{z} := \mathbf{z}'(i)$ [$\mathbf{z}$ remains PO] and return to (II).

Step 2(b): Tree of PO graphs

Generation Lemma – elaboration of part II*:

- Suppose each $\mathbf{z}'(i)$ is not PO ≡ has a PI trade.
- $\mathbf{z}$ is PO $\rightarrow$ this PI trade is not in $\mathbf{z}$ $\rightarrow$ it must involve agent i giving o to some agent, $j(i)$.
- Create a graph on $1,\ldots,k$ with edges $i \rightarrow j(i)$.
- This graph has a cycle. Concatenate all the PI trades along the cycle and get a long PI trade.
- In the long PI, each agent both receives o (from the agent before it) and gives o (to the agent after it).
- Hence, the PI trade can be done without o.
- Hence, the PI exists in $\mathbf{z}$ too – contradiction.

Step 2(c): PO+EF allocations

For each consumption graph in leaf of tree:

- If it has cycle with product < 1 – discard leaf.
- If #sharings (= #edges – m) $> n$-1 – discard leaf.
- Otherwise, create a linear program with:
  - One variable for each sharing (at most n-1).
  - Feasibility constraint for each shared good.
  - EF constraint for each pair of agents (n^2-n).
- Size of LP does not depend on m.
- Can be solved in constant time (for fixed n).
- All in all: $O(m^2)$ time for each leaf.

→ Minimal-sharing is solved in $O(\text{poly}(m))$. □

Conclusion

The minimal sharing algorithm is useful when:

- There are few agents and many objects.
- The objects are too valuable for approximate-fairness or fairness-in-expectation.
- Objects can be shared, but it is undesired.

Variants (in paper)

- Minimize the number of *shared goods* instead of the number of *sharings*.

- Use *other fairness definitions* instead of *envy-freeness*:
  - **Proportionality** := each agent's value is at least $1/n$ of the total value.
  - **Weighted proportionality** := proportionality with different entitlements.
  - **Equitability** := all agents have the same subjective value.

Variants (in paper)

If we want fairness but do not care about PO:

- Finding an allocation with ≤ n-1 sharings becomes computationally easier.
- Minimizing the number of sharings becomes *NP-hard* even when agents have generic valuations (reduction from Partition).

Ongoing and Future Work

- **Negative and mixed values**.
- **Consensus division**: each agent values each bundle as exactly $1/n$.
  - Useful for truthful allocation mechanisms.
  - Upper bound: $n*(n-1)$ sharings.
- **Non-linear sharing**, e.g.: a fraction x of an object gives only a fraction x^2 of its utility.
- **Minimize the utility of shared goods**, instead of just their number.

Thank you for coming ☺