Fair Land Reform:

How to Re-Divide Land Fairly?

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Computer Science Department

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Scotland Land Reform: Fairness vs. Ownership Rights

Fairness

"Just over 1200 landowners hold two-thirds of Scotland's land" (Bryden and Geisler, Land Use Policy, 2007)

"The Scottish Government's vision is that Scotland's land must be an asset that benefits the many, not the few... through a democratically accountable... system of land rights that promotes **fairness** and social justice..." (Scottish Government website, 2014)

Ownership Rights

"Scottish Land & Estates warned of... infringement of landowners' **ownership rights** under the European Convention on Human Rights."

(Telegraph, 20 May 2015)

"The [UK] Supreme Court... finds that... A1P1 rights [ownership rights] were violated by section 72(10) of the 2003 Act..." (Salvesen v Riddell case 2013())

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Model: Cake Cutting and Land Division

$Cake = [0, 1]$.

n agents. Agent i has value-density $v_i : Cake \to \mathbb{R}^+$.
Value = integral of density: $V_i(X) := \int_X v_i(x)dx$.

Fairness: $\forall i : V_i(X_i) \geq \frac{1}{n} \cdot V_i(Cake)$
Easy to find for any number of agents (Even & Paz, 1984).

Ownership: $\forall i : V_i(X_i) \geq V_i(X_i^0)$

In general, Fairness and Ownership are **incompatible**.

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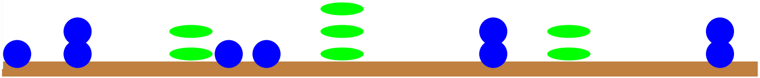

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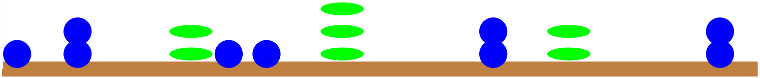

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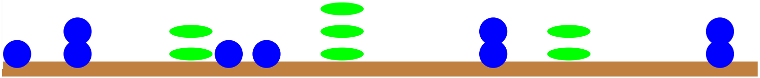

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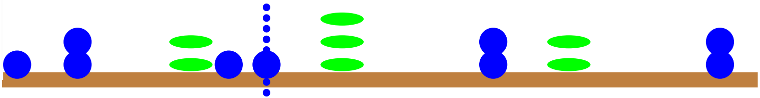

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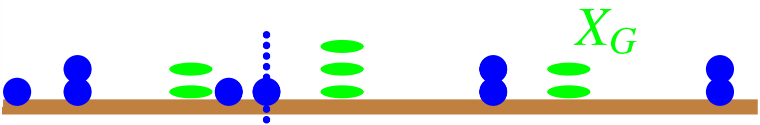

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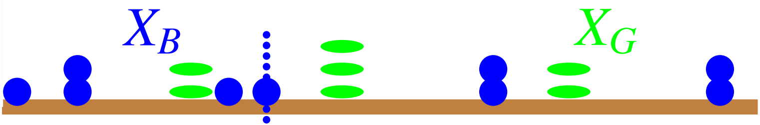

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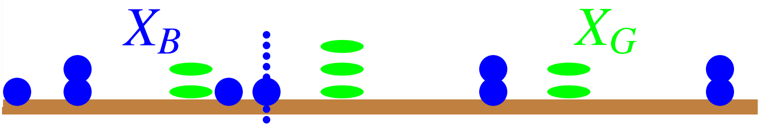

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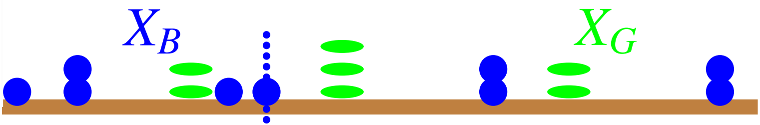

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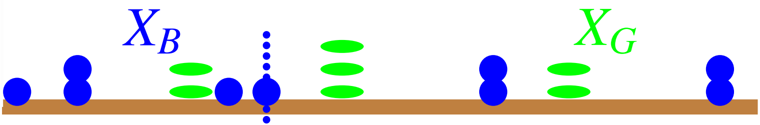

Fairness-Ownership Balance

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Theorem (1)

If $f + w = 1$, then there exists a division simultaneously satisfying w-ownership and f-fairness.

Fairness-Ownership Balance: Existence

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Proof.

The set of utilities of divisions is convex (Dubins& Spanier, 1961). Given divisions X^0, Y, whenever $f + w = 1$, there exists a division Z such that:

$$\forall i : V_i(Z_i) = w \cdot V_i(X_i^0) + f \cdot V_i(Y_i)$$

Take X^0 as original division and Y as any fair division.

Then Z satisfies w-ownership and f-fairness. □

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Problem: not constructive.

Fairness-Ownership Balance: Protocol

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Theorem (1 - constructive version)

If $f + w = 1$ and f, w are rational numbers ($w = \frac{P}{Q}$), then there is a division protocol satisfying w-ownership and f-fairness.

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- Agent j cuts $X_i^0 \cap Y_j$ to Q subjectively equal pieces.
- Agent i picks P favorite pieces (relative value at least w).
- Agent j takes remaining $Q - P$ pieces (relative value f).

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Interpretation: a land-reform protocol.

Fairness-Ownership Balance: Example

There are $n = 6$ agents. Government chooses $w = f = 1/2$.

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Problem: Land is not cake! People prefer connected plots.

Model: Cake Cutting with Connected Preferences

Initially, we assumed that value = integral of density:

$$V_i(X) := \int_X v_i(x)dx$$

Now, we assume that value = integral of density **in an interval**:

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Proposition: no combination of constant (f, w) is compatible!

Suppose we want to re-divide a cake f-fairly, for some $f > 0$. For every integer $k \in \{1, \ldots, n\}$, it is possible that in any f-fair division, k agents will receive at most k/n of their initial value.

Fairness-Ownership-Connectedness Balance

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So one agent might receive at most $1/n$ of initial value.
And $n/3$ agents might receive at most $1/3$ their initial value.

Definition: Democratic ownership

A division X satisfies **democratic-ownership** if, for every integer k, there are at least $n - k$ agents for whom:

$$V_i(X_i) \geq (k/n) \cdot V_i(X_i^0).$$

So $n - 1$ agents will receive at least $1/n$ of initial value.
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Q: is democratic-ownership compatible with f-fairness ($f > 0$)?

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Fairness-Ownership-Connectedness Protocol

Theorem (2)

*There is a protocol for finding a **connected** division satisfying **democratic-ownership** and **(1/3)-fairness**.*

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Auxiliary Protocol: Archipelago Division

Input:

- $Archipelago =$ a set of M intervals ("islands").
- N agents.

Output: a division X in which for each agent i:

- X_i is an interval entirely contained within a single island;
- $V_i(X_i) \geq V_i(Archipelago)/(M + N - 1)$

Main idea: induction on the number of islands M.

- $M = 1$: use Even &Paz on single interval.
- $M > 1$: Auction one island. Recurse on remaining islands.

Archipelago Division Protocol: Example

There are $N = 5$ agents and $M = 3$ islands.
Normalization: all agents value the entire archipelago as $M + N - 1 = 7$. We will give each agent an interval worth 1.

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Agents are sent in order to island #1, until the value of the next agent is smaller than the island population plus 1.
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We have $N' = 3$ remaining agents and $M' = M - 1 = 2$ islands.
The remaining agents value island #1 as less than $(N - N' + 1) = 3$, So they value the remaining islands as at least $(M + N - 1) - (N - N' + 1) = N' + M' - 1$.
So we can recursively divide the remaining islands to the remaining agents.

Fairness-Ownership-Connectedness Protocol

Theorem (2)

*There is a protocol for finding a **connected** division satisfying **democratic-ownership** and **(1/3)-fairness**.*

Main Protocol (2)

Input:

- Original division X^0 with n intervals.
- n agents.

For each agent i, create a 'helper' i^*, who wants only X_i^0:

$$V_{i*}(X_i^0) = 1 \qquad V_{i*}(Cake \setminus X_i^0) = 0$$

Use the Archipelago Division protocol with:

- Every piece of X^0 as an "island": $M = n$ islands overall.
- n real agents and n helpers: $N = 2n$ agents overall.

Let each agent i select either its real piece or the piece of i^*.

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Lemma. The division of Protocol (2) satisfies (1/3)-fairness.

Proof. The Archipelago Division Protocol guarantees that for each real agent i: $V_i(X_i) \geq V_i(Cake)/(M+N-1)$.
Here, $M = n$ and $N = 2n$, so $V_i(X_i) > V_i(Cake)/(3n)$.

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Lemma. The division satisfies democratic-ownership.

Proof. For every k, at most k islands will be divided among n/k or more real agents. So at least $n-k$ islands will be divided among less than n/k real agents.
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Problem: Land is not 1-D cake! It is 2-dimensional.

Fairness-Ownership-Geometry

The common cake model is **1-dimensional** (interval). But, land is (at least) **2-dimensional**.

People care about the **geometric shape** of their land-plot. E.g, some people may prefer a **rectangular** plot:

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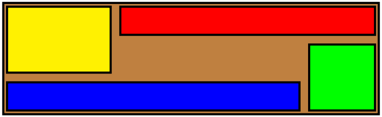
With rectangular pieces, we cannot assume that X^0 is a complete partition:



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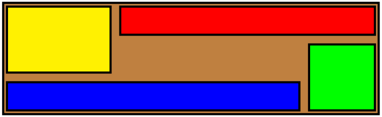

Theorem (3)

*There is a protocol for finding a **rectangular** division satisfying **democratic-ownership** and **(1/4)-fairness**.*

Proof.

Using geometric techniques, we can prove that (a) all holes are rectangular, and (b) the number of holes is less than n.
Therefore, in the Archipelago Division protocol there are at most $2n$ rectangular islands: $M < 2n$.
Therefore, the value guarantee per agent is:
$1/(M+N-1) > 1/(2n+2n-1) > 1/(4n)$. □

Conclusions

- In classic cake-cutting, **Fairness** is the main consideration.
- Land-division adds **Ownership** and **Geometry**.
- We made first steps in balancing these considerations.
- *Open question: what other considerations are important for a successful land reform?*

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Thank you!