1 Scientific Background

A *fair division problem* is a problem in which several objects have to be divided among several agents, who may have different preferences over the objects, such that all agents agree that the allocation is fair. Fair division problems vary according to the nature of the objects being divided, the preferences of the agents, and the criterion for fairness. Algorithms for solving fair division problems have been developed and used since ancient times; see, for example, the algorithm used by Abram and Lot for dividing Canaan (Genesis 13). During the last 70 years, since the seminal paper of Steinhaus [60], many new algorithms for fair division have been developed by mathematicians, economists, political scientists and computer scientists. Surveys of various such algorithms can be found at [17, 19, 20, 22, 23, 42, 45, 50, 52].

However, only in recent years are these new algorithms being employed for solving practical problems on a daily basis.

An example of such a successful employment is the problem of *fair rent division*, in which several housemates have to decide who gets which room, and (since the rooms differ in quality) how much rent each of them should pay. The division should be *envy-free*, which means that each housemate should feel that his/her bundle of room+rent is at least as good as the bundle of any other housemate. Many years of theoretical research into this problem, starting with Su [61], have led to the development of several algorithms for fair rent division. Some of these algorithms have been deployed in websites that are available for public use.[1] Users of such websites can input their preferences into the website, and in a few seconds, receive a room assignment and rent division that is *guaranteed* to be envy-free. This saves the users hours of frustrating negotiations, and lets them solve their rent disputes quickly and peacefully. Gal et al. [31] report that, as of February 2016, the rent division application in their `spliddit.org` website [34] has been used to solve over 10,000 rent-division instances. The following is one of many positive reviews that they received:

> "This tool helped us a lot. We live in a flat populated by international, young people, so its been almost a revolving door of roommates [...] With your method we were able to avoid any long discussions. Thank you."

[1] See `http://www.spliddit.org/apps/rent/` and `https://www.nytimes.com/2014/04/29/science/to-divide-the-rent-start-with-a-triangle.html` and `https://fairoutcomes.com/`.

In contrast to the success of rent division algorithms, many other fair division problems are still solved using "old school" methods such as expensive expert assessments and lengthy negotiations. The reason is that existing fair division algorithms are not sufficiently suited for the real-life requirements of these problems. Some examples are presented below.

(a) Real-estate division. Cases of inheritance, divorce and partnership dissolution involving lands and houses are usually handled by real-estate assessors. The process is expensive and time-consuming, and there is no guarantee that all partners involved will find the allocation envy-free. One would hope that land estates could be divided using algorithms for *fair cake-cutting* [20, 29, 48, 50, 60]. However, such algorithms usually assume that the cake is one-dimensional and ignore its two-dimensional geometric aspects. Dividing houses is even harder, since houses (in contrast to cakes or lands) cannot be cut. In contrast to the rent division problem, in inheritance division it is often unwanted to use monetary transfers due to tax implications. Most algorithms for fair allocation of such *indivisible* objects, e.g. [15, 16, 24, 26, 39, 43], guarantee only *approximate* fairness, for example, "envy-free up to one item". This is unacceptable when dividing highly-valuable objects such as houses.

(b) Allocation of municipal assets. Municipal governments often own a large number of buildings and other assets [36, 37]. Such assets can be used for various purposes benefitting different populations, such as: schools for children, clubs for the elderly, etc. Often, the allocation of such assets is a source of dispute among sectors in the population. For example, different sectors with different educational approaches may demand that certain buildings be allocated as schools for their children. This is again a problem of *fair indivisible object allocation*. However, in contrast to inheritance division where the agents (the heirs) are individuals, here the agents (the sectors) are *groups*. Different members in the same group may have different preferences regarding which building is better. Therefore, algorithms for fair allocation among individuals are not directly applicable in this case.

(c) Coalitional-government formation. In parliamentary democracies, such as Israel, governments are almost always formed by a coalition of parties. The formation of such governments requires to decide which party receives what ministry, as well as to balance the different requests of the parties regarding policy and legislation. Usually, these decisions may take weeks and even months of barganing and negotiations [2, 41]. A few countries, such as North Ireland and Denmark, treat this problem as a problem of *fair indivisible object allocation*, and apply a method of *sequential allocation* by which each party in turn picks a ministry, where the turns are determined by the party sizes [18, 46]. However, such methods are not common due to several reasons. First, as in the case of houses, the sequential allocation guarantees only approximate fairness. Second, as in the case of municipal assets, the ministries are effectively allocated to groups (the party voters). Different voters of the same party may have different preferences about which ministry the party should take. Third, government formation requires to decide not only about ministry allocation but also about other issues such as the policy guidelines of the government.[2]

(d) Electricity allocation. In many developing countries, electricity is a scarce resource. The supply of electricity to villages is often smaller than the demand, and it is required to periodically disconnect households from the power-plant — a process known as *load shedding*. Until recently, this process was done without much consideration of fairness aspects or the personal preferences of the households; Oluwasuji et al. [47] were the first to suggest fairness-based methods for load shedding. However, their methods are heuristic: while they perform reasonably well in simulations on Nigerian electricity consumption data, they do not provide general fairness guarantees. In contrast to other settings, here several people may be allocated the same resource (electricity) at the same time. A complicating factor is that the number of such people may change with time, depending on the demand: in low-demand hours it may be possible to serve more people simultaneously than in high-demand hours.

2 Research Objectives and Expected Significance

The present research is inspired and motivated by the success story of the algorithms for fair rent division, described in the introduction. The main research objective is to spread this

[2] A fourth aspect of the government formation problem is that the ministries are not fixed in advance: the very number and definition of the ministries are subject to negotiation. This aspect is currently beyond the scope of the present research, although it is an interesting subject for future research.

success to many other areas in which there are disputes over resource allocation.

To make the research concrete and practical, I have chosen to focus on the four specific settings outlined in the previous section: real-estate division among heirs, municipal asset allocation among groups, ministry allocation among parties, and electricity division among households. These settings raise several fundamental theoretical challenges, such as allocating discrete objects when approximate fairness is unacceptable, allocating objects among groups rather than individuals, and allocating resources when the demand changes over time. These theoretical challenges may be relevant to other settings as well. Thus, the present research may have a wider impact on various fair division problems.

In the long run, the expected significance of this research is to fundamentally change the way people handle resource allocation problems: rather than spending days, weeks or even months in arguments and disputes, they will use an algorithm to attain a provably fair solution in a quick and peaceful way.

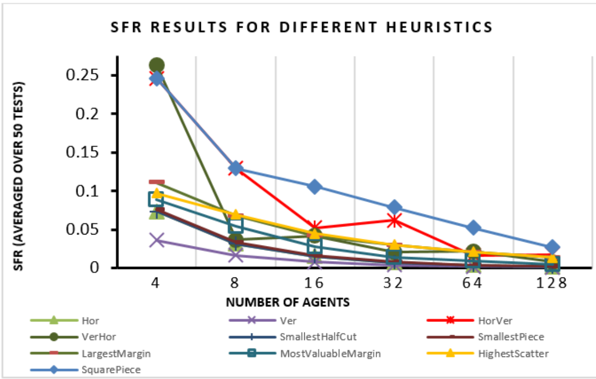
3 Detailed Description of Proposed Research

The main part of the research is theoretical, and its purpose is to cope with the mathematical challenges raised by the fair division problems described above.

3.1 Fair division of land estates

Division of land estates happens both in inheritance cases and when dividing public lands among citizens, for example when a new agricultural settlement is founded. Most fair division algorithms ignore the two-dimensional geometric shape of the land, and thus may produce very long and narrow land-plots that are unusable in practice. In previous work [52, 57, 58] I presented various algorithms that explicitly handle geometric constraints on the land-plots, for example, guarantee that the resulting plots are squares or fat rectangles. However, such algorithms may be unfit for inheritance division since they may leave some land unallocated.

In a more recent preliminary study (joint with Dr. Rica Gonen and her student, Shtechman et al. [59]), we took a different approach to two-dimensional land division: we adapted the famous *Even-Paz algorithm* [29], originally designed for a 1-dimensional cake, to a 2-



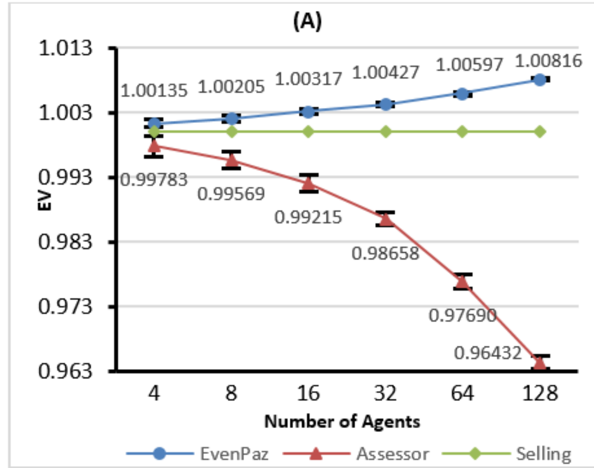


Figure 1: **Left**: comparison of the average aspect ratio of land-plots when dividing land using different cut-direction heuristics. The aspect ratio is always highest when using a heuristic that we call "SquarePiece" (see [59] for details). In contrast, in utilitarian and egalitarian welfare, the best performer is a different heuristic called "MostValuableMargin". **Right**: comparison of egalitarian welfare between the Even-Paz algorithm with the "MostValuable-Margin" heuristic and other methods for land division, namely division by an assessor and selling the land for its market value. Vertical bars denote 95% confidence intervals.

dimensional land-estate, such that all land is allocated. This adaptation requires to define the directions by which the cake is cut. While in a 1-dimensional cake each cut is a point, in a 2-dimensional cake each cut is a line, and there are infinitely many directions by which this line can be drawn on the cake. Even with the natural restriction that each cut should be either horizontal or vertical, there are still 2^{n-1} possible combinations of such directions (since with n agents there are $n-1$ cuts). In the above study, we developed 10 heuristics for determining the cut directions, and applied the algorithm with the heuristics to real land-value data maps of two different countries — New Zealand and Israel. We evaluated each heuristic on different measures, such as the *utilitarian welfare* (the sum of agents' utilities), the *egalitarian welfare* (the poorest agent's utility), the level of envy, and the aspect ratio of the resulting pieces. Some heuristics are clearly superior in each of these measures, and perform significantly better than assessor division. Example results are illustrated in Fig.1.

While the results for the heuristic methods are promising, we currently do not have an algorithm that provably optimizes each of these objectives. Such optimization problems are known to be computationally hard even for a one-dimensional cake [4, 12], but this does not preclude solving small instances exactly, or solving large instances approximately.

Research Objective 1. Develop algorithms for dividing a two-dimensional land-estate, which allocate the *entire* land, while optimizing (or approximately optimizing) measures such as utilitarian welfare, egalitarian welfare, envy level or aspect ratio.

An additional shortcoming of the preliminary study was that the land-value maps we could use were of relatively low resolution — the maps covered all of New Zealand and all of Israel, and each cell was about 500 by 500 meters long.

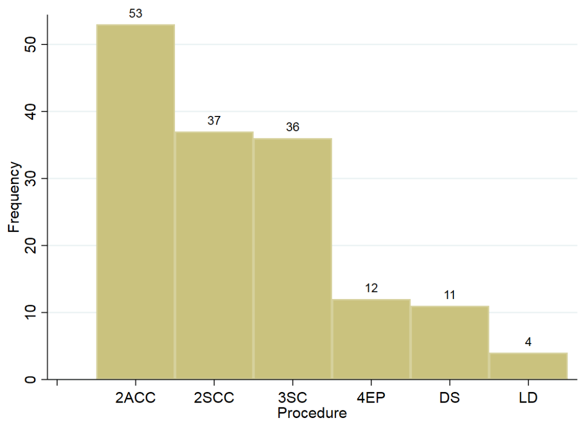
Research Objective 2. Construct land-value maps of smaller regions and higher resolutions. Test the new algorithms and compare them to existing algorithms on these maps.

In another preliminary work (joint with Dr. Josue Ortega and Maria Kyropoulou [40]), we have conducted a laboratory experiment in which we let people divide virtual "cakes" (one-dimensional lines on a computer screen) using various algorithms for cake-cutting. The experiment results provided many insights about which algorithms are considered fairer, which of them are easier to use, and which of them are easier to manipulate. Some sample results are illustrated in Figure 2; full results and explanations can be found in the paper. We plan to extend this experiment in various aspects that may make it more relevant to practical land division settings.

First, in the original experiment some subjects noted that it may be helpful to let them chat with their partners during the division process and look for a win-win solution, rather than just feed their input to the computer and wait for the output. We plan to check this suggestion by comparing three treatments: automated division without chat, automated division with option to chat, and non-automated division based on chat only. For each treatment, we plan to check the measures reported in the original experiment.

Second, the cake presented to the subjects in the original experiment was an artificial one-dimensional line. We would like to perform similar experiments where the division is done on two-dimensional maps of actual land estates. An advantage of using a real land map rather than an artificial one is that human subjects have actual preferences on which part of the land is better, so they may be able to answer questions such as "what algorithm gave you the piece that you are most satisfied with?".

Research Objective 3. Perform experiments similar to Kyropoulou et al. [40] in which



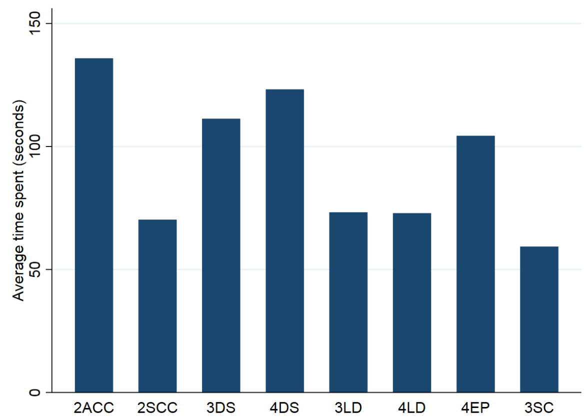


Figure 2: **Left:** percentage of subjects who listed each of several cake-cutting procedures as "fairest" (they could list more than one). The three best scoring procedures are 2ACC and 2SCC (two variants of cut-and-choose) and 3SC (the Selfridge-Conway algorithm). These algorithms differ than the other algorithms in that they guarantee *envy-freeness*, whether the others guarantee only a weaker fairness notion called *proportionality*. See [40] for details. **Right:** Average time spent in each procedure (in seconds). We consider this measure a proxy of the perceived complexity of the procedure. By this measure, the simpler procedures are 2SCC (a variant of cut-and-choose), 3LD and 4LD (variants of the "Last Diminisher" algorithm of Steinhaus [60]), and 3SC (Selfridge-Conway).

(a) subjects can chat during the division; (b) subjects divide actual maps of land-estates. In each such experiment, compare old and new algorithms for fair cake-cutting on measures such as: perceived fairness, ease of use, ease of manipulation, aspect ratio of resulting pieces, and satisfaction with the allocated plot.

3.2 Fair allocation of discrete objects

In settings in which monetary transfers are undesirable or impossible, such as inheritance division or cabinet ministry allocation, the two main approaches to fair object allocation are *approximate fairness* [5, 7, 8, 11, 15, 16, 24, 26, 32, 39, 43] and *randomized fairness* [1, 13, 14, 25, 35, 38]. I, too, have made a modest contribution to the literature on approximate fairness [3, 53, 56]. Nonetheless, I believe these approaches are inappropriate when the allocated objects are highly valuable.

Therefore, in an ongoing work with Dr. Fedor Sandomirskiy, we have began to develop a new approach to this problem: *fair division with minimal sharing*. In this approach, the goal is to attain absolute deterministic fairness by sharing some of the objects among two or more agents, while minimizing the number of shared objects. This approach formalizes some practices that are already applied in real life. In inheritance cases, when monetary transfers are undesirable due to tax implications, it is common to designate one apartment as a "remnant apartment", and allocate fractions of this apartment among the heirs so as to balance the inequality in the allocation of the other apartments.[3] For example, if four apartments have to be divided among three heirs, then each heir gets one of the three larger apartments and a certain fraction of the smallest apartment, such that an heir who gets a less desired large apartment, gets a larger fraction of the smallest apartment. Since a shared apartment is harder to use, it is desired to share as few apartments as possible. Similarly, when allocating cabinet ministries among parties, a ministry is often split into two sub-ministries in order to satisfy the fairness requests of the parties. Alternatively, the same ministry is allocated to two parties in a "rotation" agreement. Such solutions are undesired, and used only to the extent necessary for attaining fairness.

In a preliminary work [51], we formalize the objective of fair division with minimal sharing. We present an algorithm that finds an allocation that is *envy-free* and *fractionally-Pareto-optimal*, and subject to these requirements, minimizes the number of shared objects. This algorithm presents a promising alternative to current fair division algorithms. However, to make it practically useful, it should be improved in several ways.

First, the run-time complexity of this algorithm is $O(m^{\frac{n(n-1)}{2}+2})$, where m is the number of items and n is the number of agents. Since the dependence on n is exponential, the algorithm is useful only when the number of agents is small.

Research Objective 4. Develop an algorithm for fair division with minimal sharing whose running time is polynomial in both the number of agents and the number of items.

Alternatively, if this turns out to be impossible (e.g. if we find out that the problem is NP hard), then the dependence on n should be improved such that the algorithm becomes more practical for a larger number of agents.

Second, the present algorithm assumes that all agents have the same entitlements. This

[3] This description is based on personal conversations with Achikam Bitan, who has been a real-estate assessor in Israel since 1971 (http://ab-shamaim.com).

is a reasonable assumption in inheritance division, but not in cabinet ministry allocation, in which larger parties naturally have a larger entitlement. Most fair division algorithms assume that all agents have equal entitlements; only a small number of them have been adapted to agents with different entitlements [6, 8, 10, 21, 27, 30, 44, 49]. I, too, have managed to adapt some fair division results to settings with different entitlements [53, 54]. Based on these results, I believe the minimal-sharing algorithm can be adapted too.

Research Objective 5. Develop an algorithm for fair division with minimal sharing among agents with different entitlements.

3.3 Fair division among groups

Most fair division algorithms assume that each part of the allocated resource is given to an individual agent. Yet, in reality resources often have to be allocated among *groups of agents*, who use the resource together. For example, when a municipal government designates a building as a school for a certain sector of the population, all children in this sector may enjoy the building simultaneously. Naturally, different group members may have different preferences. The same share can be perceived as fair by one member and unfair by another member of the same group. Ideally, we would like to find an allocation that is considered fair by *all* agents in *all* groups. However, a number of recent works, both by me and by other authors, show that this "unanimous fairness" might be too strong to be practical. This has been shown for cake-cutting [55], indivisible object allocation [62], divisible resource allocation [9] and rent division [33].

In light of this impossibility, I recently proposed (in joint works with Prof. Shmuel Nitzan, Prof. Sophie Bade and Dr. Warut Suksompong), a new approach to fair division among groups: *democratic fairness*. In this approach, the goal is to guarantee that the allocation is fair for a certain fraction h of the members in each group. The fraction h should be as large as possible, and in particular, at least $1/2$ (so that in each group, at least a weak majority of the population supports the agreement). This is inspired by current practices in democratic societies, where most decisions are accepted by a majority voting rather than unanimously.

In preliminary works, we developed algorithms for democratically-fair allocation of continuous heterogeneous resources such as land estates [55], and democratically-approximately-fair allocation of discrete objects [56]. The new approach presented in the previous subsection — fair division with minimal sharing — has not yet been extended to allocation among groups. This extension is particularly important for the dividing municipal assets among population sectors, as well as for dividing cabinet ministries among parties.

Research Objective 6. Develop an algorithm for democratically-fair division with minimal sharing among groups of agents.

For Pareto-efficient and envy-free allocation of *homogeneous* resources, we proved that, for two groups, there exists a unanimously-fair allocation [9]. However, for three or more groups, there may be no unanimously-fair allocation, and for five or more groups, there may even be no democratically-fair allocation. Thus, the existence of democratically-fair allocations for three or more groups remains open.

Research Objective 7. Prove or disprove: there always exists a Pareto-efficient allocation of homogeneous resources among three or four groups, such that at least 1/2 of the members in each group consider the allocation envy-free.

3.4 Fair division with changing demand

Consider a village in a developing country, in which the supply of electricity is limited and cannot satisfy the total demand of all households. At each point in time, only a certain subset of the households can be connected to electricity, while the others remain in the dark (literally). The households may differ in the time in which they prefer to be connected: some households prefer to be connected in evenings, some in weekends, etc. Recently, Oluwasuji et al. [47] presented a comprehensive survey of various methods for electricity division, and presented some new heuristic methods. Their new methods attain reasonable fairness levels when simulated on Nigerian consumption patterns, but do not guarantee fairness. Inspired by their work, I have recently started working with Dinesh Kumar Beghal (a prospective Ph.D. student from Galgotias University in India) on algorithms that guarantee fairness.

In the special case in which the supply is 1, and there are n households with a constant demand of 1, the problem is equivalent to the classic *1-dimensional cake-cutting* problem, where the “cake” is the time (e.g. the 168 hours of a week). Standard cake-cutting algorithms, such as Even and Paz [29], can be used to partition the time into n pairwise-disjoint intervals, such that each household values having electricity at its allocated interval at least $1/n$ of its value for having electricity at the entire week. If the supply is $k > 1$, where k is an integer, then this algorithm can be adapted to allocate each household an interval worth at least k/n of its total value. However, this adaptation still assumes that all agents have the same demand (1), and moreover, that their demand remains constant during the week. In reality, different households may have different demands, and the demand may change over the week. Moreover, the changes in demand during the week can often be forecasted with reasonable accuracy [28]. These forecasts may potentially be used for developing new fair division algorithms that take this changing demand into account.

Research Objective 8. Develop an algorithm for fair allocation of time among agents with different demands (of electricity), where the demand may change over time.

References

[1] Atila Abdulkadiroglu and Tayfun Sonmez. Random serial dictatorship and the core from random endowments in house allocation problems. *Econometrica*, 66(3):689, 1998.

[2] Takanori Adachi and Yasutora Watanabe. Ministerial weights and government formation: Estimation using a bargaining model. *Journal of Law, Economics and Organization*, 24(1):95–119, 2008. ISSN 87566222, 14657341. URL http://www.jstor.org/stable/40058158.

[3] Elad Aigner-Horev and Erel Segal-Halevi. Envy-free matchings in bipartite graphs and their applications to fair division, 2019. arXiv preprint 1901.09527.

[4] Yonatan Aumann, Yair Dombb, and Avinatan Hassidim. Computing Socially-Efficient Cake Divisions. In *The International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 343–350, May 2013. URL http://dl.acm.org/citation.cfm?id=2484976.

[5] Haris Aziz, Gerhard Rauchecker, Guido Schryen, and Toby Walsh. Approximation algorithms for max-min share allocations of indivisible chores and goods, 2016. arXiv preprint 1604.01435.

[6] Haris Aziz, Hau Chan, and Bo Li. Maxmin share fair allocation of indivisible chores to asymmetric agents. In *Proceedings of AAMAS'19*, pages 1787–1789. IFAAMAS, 2019. arXiv preprint 1906.07602.

[7] Haris Aziz, Herve Moulin, and Fedor Sandomirskiy. A polynomial-time algorithm for computing a pareto optimal and almost proportional allocation, 2019. arXiv preprint 1909.00740.

[8] Moshe Babaioff, Noam Nisan, and Inbal Talgam-Cohen. Fair allocation through competitive equilibrium from generic incomes. In *Proceedings of the Conference on Fairness, Accountability, and Transparency*, pages 180–180. ACM, 2019. URL http://www.inbaltalgam.com/papers/Fair\%20Allocation\%20through\%20CEGI.pdf. arXiv preprint 1703.08150 v2 (2018-09).

[9] Sophie Bade and Erel Segal-Halevi. Fair and efficient division among families, 2018. arXiv preprint 1811.06684.

[10] Julius B. Barbanel. Game-theoretic algorithms for fair and strongly fair cake division with entitlements. In *Colloquium Mathematicae*, volume 69, pages 59–73, 1996.

[11] Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. Finding fair and efficient allocations. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, pages 557–574. ACM, 2018. URL http://arxiv.org/abs/1707.04731. arXiv preprint 1707.04731.

[12] Xiaohui Bei, Ning Chen, Xia Hua, Biaoshuai Tao, and Endong Yang. Optimal Proportional Cake Cutting with Connected Pieces. In *Proceedings of the 26th AAAI Conference on Artificial Intelligence (AAAI-12)*, pages 1263–1269, 2012. URL `http://www.aaai.org/ocs/index.php/AAAI/AAAI12/paper/download/4980/5456`.

[13] Anna Bogomolnaia and Herve Moulin. A New Solution to the Random Assignment Problem. *Journal of Economic Theory*, 100(2):295–328, October 2001. ISSN 00220531. doi: 10.1006/jeth.2000.2710. URL `http://dx.doi.org/10.1006/jeth.2000.2710`.

[14] Anna Bogomolnaia, Herve Moulin, and Fedor Sandomirskiy. A simple online fair division problem, 2019.

[15] Sylvain Bouveret and Michel Lemaître. Characterizing conflicts in fair division of indivisible goods using a scale of criteria. *Autonomous Agents and Multi-Agent Systems*, 30 (2):1–32, 2016.

[16] Sylvain Bouveret, Yann Chevaleyre, and Nicolas Maudet. Fair Allocation of Indivisible Items. In Felix Brandt, Vincent Conitzer, Ulle Endriss, Jerome Lang, and Ariel D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 13, pages 284–310. Cambridge University Press, 2016. URL `http://www.cs.cmu.edu/\~{}arielpro/papers/cakechapter.pdf`.

[17] Steven J. Brams. *Mathematics and Democracy: Designing Better Voting and Fair-Division Procedures*. Princeton University Press, 1st edition, December 2007. ISBN 0691133212.

[18] Steven J. Brams and Todd R. Kaplan. Dividing the Indivisible: Procedures for allocating cabinet ministries to political parties in a parliamentary system. *Journal of Theoretical Politics*, 16(2):143–173, 2004. doi: 10.1177/0951629804041118.

[19] Steven J. Brams and Christian Klamler. Fair Division. In Robert A. Meyers, editor, *Encyclopedia of Complexity and Systems Science*, pages 1–12. Springer Berlin Heidelberg, Berlin, Heidelberg, 2017. ISBN 978-3-642-27737-5.

[20] Steven J. Brams and Alan D. Taylor. *Fair Division: From Cake Cutting to Dispute Resolution*. Cambridge University Press, Cambridge UK, February 1996. ISBN 0521556449.

[21] Steven J. Brams, Michael A. Jones, and Christian Klamler. Divide-and-Conquer: A Proportional, Minimal-Envy Cake-Cutting Algorithm. *SIAM Rev.*, 53(2):291–307, May 2011. doi: 10.1137/080729475. URL `http://drops.dagstuhl.de/volltexte/2007/1221/pdf/07261.BramsSteven.Paper.1221.pdf`.

[22] Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia, editors. *Handbook of Computational Social Choice*. Cambridge University Press, 1 edition, April 2016. ISBN 1107060435. URL `http://www.worldcat.org/isbn/1107060435`.

[23] Simina Brânzei. *Computational Fair Division*. PhD thesis, Faculty of Science and Technology in Aarhus university, 2015.

[24] Eric Budish. The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes. *Journal of Political Economy*, 119(6):1061–1103, December 2011. ISSN 00223808. doi: 10.1086/664613.

[25] Eric Budish, Yeon-Koo Che, Fuhito Kojima, and Paul Milgrom. Designing random allocation mechanisms: Theory and applications. *American Economic Review*, 103(2): 585–623, 2013.

[26] Ioannis Caragiannis, David Kurokawa, Herve Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. The Unreasonable Fairness of Maximum Nash Welfare. In *Proceedings of EC'16*. ACM Press, 2016.

[27] Ágnes Cseh and Tamás Fleiner. The complexity of cake cutting with unequal shares. In *Proceedings of SAGT'18*, pages 19–30. Springer, 2018. arXiv preprint 1709.03152.

[28] Linh Phuong Catherine Do, Kuan-Heng Lin, and Peter Molnár. Electricity consumption modelling: A case of germany. *Economic Modelling*, 55:92–101, 2016.

[29] Shimon Even and Azaria Paz. A Note on Cake Cutting. *Discrete Applied Mathematics*, 7(3):285–296, March 1984. ISSN 0166218X. doi: 10.1016/0166-218x(84)90005-2. URL `http://dx.doi.org/10.1016/0166-218x(84)90005-2`.

[30] Alireza Farhadi, Mohammad Ghodsi, Mohammad Taghi Hajiaghayi, Sebastien Lahaie, David Pennock, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. Fair allocation of indivisible goods to asymmetric agents. *Journal of Artificial Intelligence Research*, 64: 1–20, 2019. arXiv preprint 1703.01649.

[31] Yaakov Kobi Gal, Moshe Mash, Ariel D Procaccia, and Yair Zick. Which is the fairest (rent division) of them all? *Journal of the ACM (JACM)*, 64(6):39, 2017.

[32] Mohammad Ghodsi, MohammadTaghi HajiAghayi, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. Fair allocation of indivisible goods: Improvements and generalizations. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, pages 539–556. ACM, 2018. arXiv preprint 1704.00222.

[33] Mohammad Ghodsi, Mohamad Latifian, Arman Mohammadi, Sadra Moradian, and Masoud Seddighin. Rent division among groups. In *International Conference on Combinatorial Optimization and Applications*, pages 577–591. Springer, 2018.

[34] Jonathan Goldman and Ariel D. Procaccia. Spliddit: Unleashing Fair Division Algorithms. *SIGecom Exch.*, 13(2):41–46, January 2015. ISSN 1551-9031. doi: 10.1145/2728732.2728738. URL `http://dx.doi.org/10.1145/2728732.2728738`.

[35] Aanund Hylland and Richard Zeckhauser. The efficient allocation of individuals to positions. *Journal of Political economy*, 87(2):293–314, 1979.

[36] Pierre W Jolicoeur and James T Barrett. Coming of age: Strategic asset management in the municipal sector. *Journal of Facilities Management*, 3(1):41–52, 2005.

[37] Olga Kaganova and Ritu Nayyar-Stone. Municipal real property asset management: An overview of world experience, trends and financial implications. *Journal of Real estate portfolio management*, 6(4):307–326, 2000.

[38] Onur Kesten and M Utku Ünver. A theory of school-choice lotteries. *Theoretical Economics*, 10(2):543–595, 2015.

[39] David Kurokawa, Ariel D Procaccia, and Junxing Wang. Fair enough: Guaranteeing approximate maximin shares. *Journal of the ACM (JACM)*, 65(2):8, 2018.

[40] Maria Kyropoulou, Josué Ortega, and Erel Segal-Halevi. Fair cake-cutting in practice. In *Proceedings EC'19*, pages 547–548. ACM, 2019. arXiv preprint 1810.08243.

[41] Michael Laver, Scott de Marchi, and Hande Mutlu. Negotiation in legislatures over government formation. *Public Choice*, 147(3/4):285–304, 2011. ISSN 00485829, 15737101. URL `http://www.jstor.org/stable/41483661`.

[42] Claudia Lindner and Jörg" Rothe. Cake-cutting: Fair division of divisible goods. In Jörg Rothe, editor, *Economics and Computation: An Introduction to Algorithmic Game Theory, Computational Social Choice, and Fair Division*, pages 395–491. Springer Berlin Heidelberg, Berlin, Heidelberg, 2016. ISBN 978-3-662-47904-9. URL `https://doi.org/10.1007/978-3-662-47904-9\_7`.

[43] R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM conference on Electronic commerce*, EC '04, pages 125–131, New York, NY, USA, 2004. ACM. ISBN 1-58113-771-0. doi: 10.1145/988772.988792. URL `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.114.363\&\#38;rep=rep1\&\#38;type=pdf`.

[44] Kevin McAvaney, Jack M. Robertson, and William A. Webb. Ramsey partitions of integers and fair divisions. *Combinatorica*, 12(2):193–201, 1992. URL `http://dx.doi.org/10.1007/bf01204722`.

[45] Hervé Moulin. *Fair Division and Collective Welfare*. The MIT Press, August 2004. ISBN 0262633116.

[46] Brendan O'Leary, Bernard Grofman, and Jørgen Elklit. Divisor methods for sequential portfolio allocation in multi-party executive bodies: Evidence from northern ireland and denmark. *American Journal of Political Science*, 49(1):198–211, 2005.

[47] Olabambo I Oluwasuji, Obaid Malik, Jie Zhang, Sarvapali D Ramchurn, et al. Algorithms for fair load shedding in developing countries. In *IJCAI*, pages 1590–1596, 2018.

[48] Ariel D. Procaccia. Cake Cutting Algorithms. In Felix Brandt, Vincent Conitzer, Ulle Endriss, Jerome Lang, and Ariel D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 13, pages 261–283. Cambridge University Press, 2015. URL `http://procaccia.info/papers/cakechapter.pdf`.

[49] Jack M. Robertson and William A. Webb. Extensions of Cut-and-Choose Fair Division. *Elemente der Mathematik*, 52(1):23–30, 1997. URL `http://dx.doi.org/10.1007/s000170050007`.

[50] Jack M. Robertson and William A. Webb. *Cake-Cutting Algorithms: Be Fair if You Can*. A K Peters/CRC Press, first edition, July 1998. ISBN 1568810768.

[51] Fedor Sandomirskiy and Erel Segal-Halevi. Fair division with minimal sharing, 2019. arXiv preprint 1908.01669. Submitted to SODA 2020.

[52] Erel Segal-Halevi. *Fair Division of Land*. PhD thesis, Bar Ilan University, Computer Science Department, 2017. Guided by Yonatan Aumann and Avinatan Hassidim.

[53] Erel Segal-Halevi. Competitive Equilibrium For almost All Incomes. In *Proceedings of AAMAS'18*, 2018. arXiv preprint 1705.04212.

[54] Erel Segal-Halevi. Cake-Cutting with Different Entitlements: How Many Cuts are Needed? *Journal of Mathematical Analysis and Applications*, 480(1):123382, 2019. doi: 10.1016/j.jmaa.2019.123382. arXiv preprint 1803.05470.

[55] Erel Segal-Halevi and Shmuel Nitzan. Fair Cake-cutting among Families. *Social Choice and Welfare*, 2019. doi: 10.1007/s00355-019-01210-9. arXiv preprint 1510.03903.

[56] Erel Segal-Halevi and Warut Suksompong. Democratic Fair Allocation of Indivisible Goods. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence (IJCAI '18)*, September 2018. URL `http://arxiv.org/abs/1709.02564`. arXiv preprint 1709.0256.

[57] Erel Segal-Halevi, Shmuel Nitzan, Avinatan Hassidim, and Yonatan Aumann. Fair and square: Cake-cutting in two dimensions. *Journal of Mathematical Economics*, 70:1–28, February 2017. ISSN 03044068. doi: 10.1016/j.jmateco.2017.01.007. URL `http://dx.doi.org/10.1016/j.jmateco.2017.01.007`.

[58] Erel Segal-Halevi, Avinatan Hassidim, and Yonatan Aumann. Envy-Free Division of Land. *Mathematics of Operations Research*, page forthcoming, 2019. arXiv preprint 1609.03938.

[59] Itay Shtechman, Rica Gonen, and Erel Segal-Halevi. Fair cake-cutting with real land value data, 2019. Submitted to AAAI 2020.

[60] Hugo Steinhaus. The problem of fair division. *Econometrica*, 16(1):101–104, jan 1948. URL `http://www.jstor.org/stable/1914289`.

[61] Francis E. Su. Rental Harmony: Sperner's Lemma in Fair Division. *The American Mathematical Monthly*, 106(10):930–942, December 1999. ISSN 00029890. doi: 10.2307/2589747. URL `http://www.math.hmc.edu/\~{}su/papers.dir/rent.pdf`.

[62] Warut Suksompong. Approximate maximin shares for groups of agents. *Mathematical Social Sciences*, 2018. http://doi.org/10.1016/j.mathsocsci.2017.09.004.